Reduced basis modeling of complex flow systems

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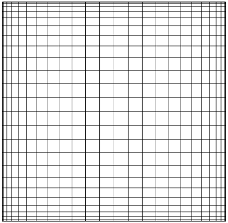
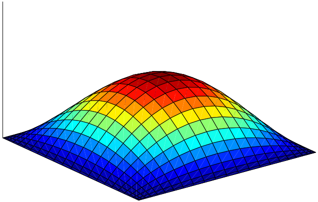
History

- Cand.Scient, UiO 2002:
  Saddle point problem - approximation of scattered data.
  Used the Morley element to achieve H^1 continuity.
- Dr.Scient, NTNU 2002-2005:
  The reduced basis element method.
  Application to complex flow systems.
  Implemented with the spectral element method.
  Supervisor: Einar Rønquist.
- Post.Doc, NTNU/Simula, 2006-:
  The reduced basis element method.
  Global C^1 maps.

The reduced basis method



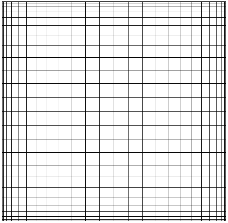


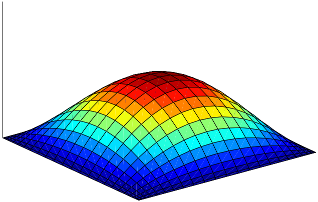
Main idea

Assume a parameter dependent problem

$$F(u;\mu) = 0.$$

The reduced basis method





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Assume a parameter dependent problem

$$F(u; \mu) = 0.$$

For small changes in the parameter μ, the corresponding solution u often varies in a smooth fashion.
We construct a set of basis functions for the problem where each basis function has a large information content.

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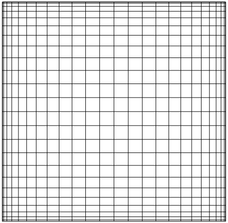

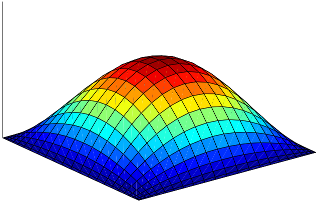

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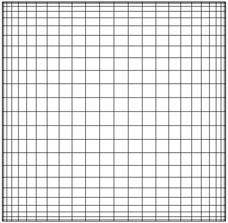

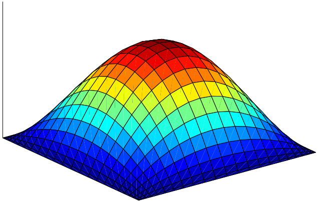

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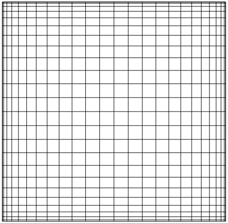

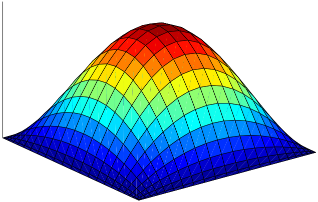

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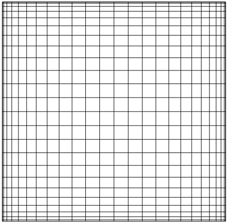

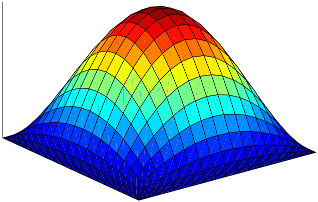

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The reduced basis method

Parameter dependent problems: $F(u;\mu) = 0$

Given $\mu \in \mathcal{D} \subset \mathbb{R}^p$, find $u \in X_{\mathcal{N}}$ such that

$$a(u, v; \mu) = \ell(v) \quad \forall\, v \in X_{\mathcal{N}}, \quad \mathcal{N} \gg 1$$

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Reduced basis procedure

Offline:

Basis functions defined on $X_{\mathcal{N}}$ span the reduced basis approximation space

$$X_N = span\{u_i\}_{i=1}^N, \quad N \ll \mathcal{N}$$

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Online:

Find the reduced basis approximation:

$$u_N(\mu) = \sum_{i=1}^{N} \alpha_i(\mu) u_i,$$

such that

$$a(u_N, v; \mu) = \ell(v) \quad \forall\, v \in X_N$$

The reduced basis method

Offline/online decoupling

- GOAL: Avoid online computations on the underlying FEM basis.
- All basis functions in X_N are stored in the FEM basis, so both $a(u_N, v; \mu)$ and $\ell(v)$ involve computations on the FEM basis.
- $\ell(v)$ is independent of the parameter, so this can be done offline.

Interpolation

Assume a basis for $X = \text{span}\{\ell_i\}_{i=1}^{N}$.
For a given v, find its projection in X, such that

$$v_X = \sum_{i=1}^{N} \alpha_i \ell_i$$

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Orthogonal basis: Diagonal l.h.s. matrix.
FEM basis: Tridiagonal l.h.s. matrix.
SEM basis: Full l.h.s. matrix.

Empirical Interpolation

Assume a basis for $X = \text{span}\{\ell_i\}_{i=1}^{N}$.
Modify each basis function: $\{\tilde{\ell}_i\}_{i=1}^{N}$,
define the "magic points" t_i For a given v, find its projection in X, such that

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$$\begin{bmatrix} \tilde{\ell}_1(t_1) & \cdots & \tilde{\ell}_N(t_1) \\ \vdots & \ddots & \vdots \\ \tilde{\ell}_1(t_N) & \cdots & \tilde{\ell}_N(t_N) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} v(t_1) \\ \vdots \\ v(t_N) \end{bmatrix}$$

Only need to evaluate v in one single point to find $v(t_i)$.
Considerable speed-up compared to global inner-product.

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Key features

- Geometry as a parameter
- Output bounds
- Offline/online decoupling
- Building blocks
- Lagrange multipliers

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Future work/ongoing work

- *A posteriori* bounds on multi-block geometries
- Three dimensional domains
- Time dependent problems
- Fluid-structure interaction

Thank you!