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Uniform preconditioners for the time dependent Stokes problem

Kent–Andre Mardal[1], **Ragnar Winther**[2]

[1] Department of Scientific Computing, Simula Research Laboratory and Department of Informatics, University of Oslo, P.O. Box 134, NO-134 Lysaker, Norway, e-mail: `kent-and@simula.no`

[2] Centre of Mathematics for Applications and Department of Informatics, University of Oslo, P.O. Box 1053 Blindern, 0316 Oslo, Norway, e-mail: `rwinther@ifi.uio.no`

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Summary Implicit time stepping procedures for the time dependent Stokes problem lead to stationary singular perturbation problems at each time step. These singular perturbation problems are systems of saddle point type, which formally approach a mixed formulation of the Poisson equation as the time step tends to zero. Preconditioners for discrete analogous of these systems are discussed. The preconditioners uses standard positive definite elliptic preconditioners as building blocks and lead to condition numbers which are bounded uniformly with respect to the time step and the spatial discretization. The construction of the discrete preconditioners is related to the mapping properties of the corresponding continuous system.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$, with n=2 or 3, be a bounded polygonal domain with boundary $\partial\Omega$. Consider the corresponding initial value problem for the time dependent Stokes problem given by:

$$\begin{aligned} \boldsymbol{u}_t - \boldsymbol{\Delta u} - \mathbf{grad}\, p = \boldsymbol{f} \quad & \text{in } \Omega \times \mathbb{R}^+, \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } \Omega \times \mathbb{R}^+, \\ \boldsymbol{u} = 0 \quad & \text{on } \partial\Omega \times \mathbb{R}^+, \\ \boldsymbol{u} = \boldsymbol{u}_0 \quad & \text{on } \Omega \times \{t = 0\}. \end{aligned} \tag{1}$$

If this initial value problem is discretized by an implicit time stepping procedure we are lead to stationary singular perturbation problems of the form:

$$\begin{aligned} (\boldsymbol{I} - \varepsilon^2 \boldsymbol{\Delta})\boldsymbol{u} - \mathbf{grad}\, p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

Here $\varepsilon > 0$ is the square root of the time step, while p is related to the original pressure by a scaling with the factor ε^2. The new right–hand side $\boldsymbol{f}$ represents a combination of the velocity at the previous time step and the original forcing term. When such implicit time stepping procedures are combined with a finite element discretization of the spatial variables, then at least one discrete analog of the system (2) has to be solved for each time step. Hence, the efficiency of such solution strategies may depend critically on the development of iterative solvers for discretizations of systems of the form (2).

We recall that many semi–implicit time stepping schemes for the full nonlinear, incompressible Navier–Stokes equation use discrete analogs of linear systems of the form (2) as building blocks. These systems are then combined with a proper method for the nonlinear convection process by a fractional step strategy, cf. for example [12], or by an approach using Lagrangian coordinates "to remove" the convective term, cf. [26].

The main purpose of the present paper is to discuss block diagonal preconditioners for discrete analogs of the system (2) when the perturbation parameter ε is allowed to be arbitrary small. More precisely, we shall assume that $\varepsilon \in (0, 1]$, and our goal is to design precondioners which lead to condition numbers that are bounded uniformly with respect to both ε and the discretization parameter h.

We note that when ε is not too small the system (2) is similar to the stationary Stokes problem, but with an additional lower order term. However, if ε approaches zero then the system formally tends to a mixed formulation of the Poisson equation. This observation can potentially indicate some problems for the corresponding discrete systems, since standard stable finite elements for the Stokes problem may not be stable for the mixed Poisson system. On the other hand, most stable elements for the mixed Poisson system, like the Raviart–Thomas elements, lack some of the continuity conditions required for conforming approximations of the Stokes system. In fact, this issue was discussed in great detail in [21], where systems of the form (2) was motivated as models for "averaged fluid flow." It was established that if standard $\boldsymbol{H}(\mathrm{div}) \times L^2$ norms was used for the mixed Poisson system then none of the most common Stokes elements appeared to be stable, and as consequence, these elements did not perform well for ε small. Motivated by this observation a new family of finite elements were constructed, with stability properties which are uniform with respect to the perturbation parameter ε.

However, for our study here the starting point is rather different. When the system (2) is derived from implicit time stepping procedures for the time dependent Stokes system the parameter ε is not a physical parameter. Difficulties which may occur as a consequence of ε being small should therefore be seen as instabilities created by the time stepping procedure, and not as instabilities created by the spatial discretizations of the time dependent Stokes system. We shall therefore in this paper study preconditioners for the system (2), discretized by standard Stokes elements. Uniform preconditioners with respect to ε and h will be derived. Standard Stokes preconditioners perform well if $\varepsilon > 0$ is sufficiently large, but degenerate if ε is small, i.e. as we approach a simple potential flow. The main tool for deriving the uniform preconditioners will be proper stability estimates in ε–dependent norms, but these norms do not degenerate to the norm of $\boldsymbol{H}(\mathrm{div}) \times L^2$ for $\varepsilon = 0$. Instead, the norm for the reduced case will correspond to $(\boldsymbol{u}, p) \in \boldsymbol{L}^2 \times H^1$.

A block diagonal preconditioner of the form studied here, where each block is composed of preconditioners for standard elliptic problems, was introduced already in [10]. Later works which study block preconditioners for discrete analogs of the system (2), using various tools of analysis, are for example [6], [13], [14], [19], and [31]. Other appraoches to the construction of preconditioners for saddle point systems are for example given in [5], [7], [20], [25], [33], and [34], cf. also [23]. The main contribution of the present paper is to relate the construction of the preconditioner for the discrete systems to the mapping properties of the continuous system (2). This leads to a clean and precise framework of analysis which is essentially independent of how the different elliptic preconditioners are constructed. All we essentially need to verify is that the discrete spaces satisfy an inf–sup condition which is uniform with respect to perturbation paramer ε.

In §2 below we introduce some useful notation and describe basic properties of the system (2). Uniform preconditioners for the continuous system is derived in §3 as a consequence of a uniform inf–sup property. In §4 we then use numerical experiments to study discrete versions of this preconditioner for some choices of finite element spaces with continuous pressures. We observe that these preconditioners seems to result in preconditioned systems which are well conditioned, uniformly with respect to the perturbation parameter ε and the mesh parameter h. In §5 we perform a detailed theoretical analysis of these discrete preconditioners. Finally, in §6 we investigate the properties of related preconditioners for finite elements with discontinuous pressures.

2 Preliminaries

For any Banach space X the associated norm will be denoted $\|\cdot\|_X$. If $H^m = H^m(\Omega)$ is the Sobolev space of functions on Ω with m derivatives in $L^2 = L^2(\Omega)$ we use the simpler notation $\|\cdot\|_m$ instead of $\|\cdot\|_{H^m}$. The space H_0^m is the closure in H^m of $C_0^\infty = C_0^\infty(\Omega)$. The dual space of H_0^m with respect to the L^2 inner product will be denoted by H^{-m}. Furthermore, L_0^2 will denote the space of L^2 functions with mean value zero. A space written in boldface denotes a n–vector valued analog of the corresponding scalar space, where n=2 or 3. The notation $(\cdot,\cdot)$ is used to denote the L^2 inner product on scalar, vector, and matrix valued functions, and to denote the duality pairing between H_0^m and H^{-m}. The gradient of a vector field $\boldsymbol{v}$ is denoted $\boldsymbol{D}\boldsymbol{v}$, i.e. $\boldsymbol{D}\boldsymbol{v}$ is the $n \times n$ matrix with elements

$$(\boldsymbol{D}\boldsymbol{v})_{i,j} = \partial \boldsymbol{v}_i / \partial x_j \quad 1 \leq i,j \leq n.$$

Hence, for any $\boldsymbol{u} \in \boldsymbol{H}^2$ and $\boldsymbol{v} \in \boldsymbol{H}_0^1$ we have

$$-(\boldsymbol{\Delta u}, \boldsymbol{v}) = (\boldsymbol{D}\boldsymbol{u}, \boldsymbol{D}\boldsymbol{v}) \equiv \int_\Omega \boldsymbol{D}\boldsymbol{u} : \boldsymbol{D}\boldsymbol{v}\, dx,$$

where the colon denotes the scalar product of matrix fields.

Below we shall encounter the intersection and sum of Hilbert spaces. We therefore recall the basic definitions of these concepts. If X and Y are Hilbert spaces, both continuously contained in some larger Hilbert spaces, then the intersection $X \cap Y$ and the sum $X + Y$ are themselves Hilbert spaces with the norms

$$\|z\|_{X\cap Y} = (\|z\|_X^2 + \|z\|_Y^2)^{1/2}$$

and

$$\|z\|_{X+Y} = \inf_{\substack{z=x+y \\ x\in X,\, y\in Y}} (\|x\|_X^2 + \|y\|_Y^2)^{1/2}.$$

Furthermore, if $X \cap Y$ is dense in both X and Y then

$$(X \cap Y)^* = X^* + Y^*. \tag{3}$$

Finally, if T is a bounded linear operator mapping X_1 to Y_1 and X_2 to Y_2, respectively, then

$$T \in \mathcal{L}(X_1 \cap X_2, Y_1 \cap Y_2) \cap \mathcal{L}(X_1 + X_2, Y_1 + Y_2).$$

In particular, we will later use the bound

$$\|T\|_{\mathcal{L}(X_1+X_2,Y_1+Y_2)} \le \max(\|T\|_{\mathcal{L}(X_1,Y_1)}, \|T\|_{\mathcal{L}(X_2,Y_2)}). \qquad (4)$$

We refer to [4, Chapter 2] for these results.

Throughout this paper $\varepsilon \in (0,1]$, $a_\varepsilon(\cdot,\cdot) : \boldsymbol{H}^1 \times \boldsymbol{H}^1 \mapsto \mathbb{R}$ will denote the bilinear form

$$a_\varepsilon(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u},\boldsymbol{v}) + \varepsilon^2(\boldsymbol{D}\boldsymbol{u},\boldsymbol{D}\boldsymbol{v}),$$

and $\boldsymbol{I} - \varepsilon^2\boldsymbol{\Delta} : \boldsymbol{H}_0^1 \mapsto \boldsymbol{H}^{-1}$ the corresponding operator, i.e.

$$((\boldsymbol{I} - \varepsilon^2\boldsymbol{\Delta})\boldsymbol{u},\boldsymbol{v}) = a_\varepsilon(\boldsymbol{u},\boldsymbol{v}) \qquad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}_0^1.$$

A weak formulation of problem (2), slightly generalized to allow for a nonhomogeneous right hand side in the second equation, is given by:

Find $(\boldsymbol{u},p) \in \boldsymbol{H}_0^1 \times L_0^2$ such that

$$\begin{aligned} a_\varepsilon(\boldsymbol{u},\boldsymbol{v}) + (p, \operatorname{div}\boldsymbol{v}) &= (\boldsymbol{f},\boldsymbol{v}) && \forall \boldsymbol{v} \in \boldsymbol{H}_0^1, \\ (\operatorname{div}\boldsymbol{u}, q) &= (g,q) && \forall q \in L_0^2. \end{aligned} \qquad (5)$$

Here we assume that data $(\boldsymbol{f}, g)$ is given in $\boldsymbol{H}^{-1} \times L_0^2$.

The problem (5) has a unique solution $(\boldsymbol{u},p) \in \boldsymbol{H}_0^1 \times L_0^2$. This follows from standard results for Stokes problem, cf. for example [15]. However, the bound on $(\boldsymbol{u},p) \in \boldsymbol{H}_0^1 \times L_0^2$ will degenerate as ε tends to zero. In fact, for the reduced problem (5), with $\varepsilon = 0$ and the boundary condition modified such that only zero normal component is required, the space $\boldsymbol{H}_0^1 \times L_0^2$ is not a proper function space for the solution. However, the theory developed in [8] can be applied in this case if we seek $(\boldsymbol{u},p)$ either in $\boldsymbol{H}_0(\operatorname{div}) \times L_0^2$ or in $\boldsymbol{L}^2 \times (H^1 \cap L_0^2)$, and with data $(\boldsymbol{f}, g)$ in the proper dual spaces. These results are in fact consequences of standard results for the Poisson equation. Here the space $\boldsymbol{H}_0(\operatorname{div})$ denotes the set of square integrable vector fields, with a square integrable divergence, and with zero normal component on the boundary.

The fact that the regularity of the solution is changed when ε becomes zero strongly suggests that ε–dependent norms and function spaces are required in order to obtain stability estimates independent of ε. Furthermore, since the reduced problem is well posed for two completely different choices of function spaces, this indicates that there are at least two different choices of ε–dependent norms. These are the norms of the spaces $(\boldsymbol{H}_0(\operatorname{div}) \cap \varepsilon \cdot \boldsymbol{H}_0^1) \times L_0^2$ and $(\boldsymbol{L}^2 \cap \varepsilon \cdot \boldsymbol{H}_0^1) \times ((H^1 \cap L_0^2) + \varepsilon^{-1} \cdot L_0^2)$. Note that for any $\varepsilon > 0$ both these spaces are equal to $\boldsymbol{H}_0^1 \times L_0^2$ as a set, but as ε approaches zero the corresponding norms degenerates to the norm of $\boldsymbol{H}_0(\mathrm{div}) \times L^2_0$ or $\boldsymbol{L}^2 \times (H^1 \cap L^2_0)$, respectively.

In [21] it was established that most standard Stokes elements are not uniformly stable in the norm of $(\boldsymbol{H}_0(\mathrm{div}) \cap \varepsilon \cdot \boldsymbol{H}^1_0) \times L^2_0$. Therefore, if we want uniform stability estimates for such elements it seems more natural to use the norm induced by the space

$$\boldsymbol{X}_\varepsilon := (\boldsymbol{L}^2 \cap \varepsilon \cdot \boldsymbol{H}^1_0) \times ((H^1 \cap L^2_0) + \varepsilon^{-1} \cdot L^2_0).$$

This is the approach taken in this paper.

The norm of the space $\boldsymbol{L}^2 \cap \varepsilon \cdot \boldsymbol{H}^1_0$, will be denoted $|\!|\!| \cdot |\!|\!|_\varepsilon$, i.e.

$$|\!|\!| \boldsymbol{v} |\!|\!|^2_\varepsilon = \|\boldsymbol{v}\|^2_0 + \varepsilon^2 \|\boldsymbol{D}\boldsymbol{v}\|^2_0,$$

while the norm in $(H^1 \cap L^2_0) + \varepsilon^{-1} \cdot L^2_0$ will be simplified to $|\cdot|_\varepsilon$. More precisely, we define, $|q|_\varepsilon$ by

$$|q|_\varepsilon = \inf_{\substack{q = q_1 + q_2 \\ q_1 \in H^1 \cap L^2_0,\, q_2 \in L^2_0}} (\|\,\mathbf{grad}\, q_1\|^2_0 + \varepsilon^{-2}\|q_2\|^2_0)^{1/2}.$$

This notation for the norm in $(H^1 \cap L^2_0) + \varepsilon^{-1} \cdot L^2_0$ is convenient, but slightly unusual, since $|q|_0 = \|\,\mathbf{grad}\, q\|_0$ is equivalent to $\|q\|_1$ on $H^1 \cap L^2_0$, while $|q|_1$ is equivalent to $\|q\|_0$.

Let $H^* \supset L^2_0$ denote the dual space of $H^1 \cap L^2_0$, and define the operator $(I - \varepsilon^2 \Delta)^{-1} : H^* \mapsto H^1 \cap L^2_0$ by a standard weak formulation, i.e. $p = (I - \varepsilon^2 \Delta)^{-1} g$ if p satisfies

$$(p, q) + \varepsilon^2(\mathbf{grad}\, p, \mathbf{grad}\, q) = (g, q) \qquad \forall q \in H^1 \cap L^2_0.$$

By using this operator a more explicit characterization of $|q|_\varepsilon$ can be given. For $q \in L^2_0$ let $q_1 = (I - \varepsilon^2 \Delta)^{-1} q \in H^1 \cap L^2_0$. Note, in particular, that $\Delta q_1 \in L^2_0$. Furthermore, a straightforward computation shows that the solution of the minimization problem in the definition of $|q|_\varepsilon$ is given by q_1 and $q_2 = -\varepsilon^2 \Delta q_1 = q - q_1$. Hence, we obtain

$$|q|^2_\varepsilon = \|\,\mathbf{grad}\, q_1\|^2_0 + \varepsilon^{-2}\|q - q_1\|^2_0. \tag{6}$$

The system (5) can alternatively be written as

$$\mathcal{A}_\varepsilon \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ g \end{pmatrix}, \tag{7}$$

where the coefficient operator $\mathcal{A}_\varepsilon$ is given by

$$\mathcal{A}_\varepsilon = \begin{pmatrix} \boldsymbol{I} - \varepsilon^2 \boldsymbol{\Delta} & -\mathbf{grad} \\ \mathrm{div} & 0 \end{pmatrix}. \tag{8}$$

Here $-\mathbf{grad} : L_0^2 \mapsto \boldsymbol{H}^{-1}$ is the dual of the divergence operator, $\operatorname{div} : \boldsymbol{H}_0^1 \mapsto L_0^2$.

Let $\boldsymbol{X}_\varepsilon^*$ be the dual space of $\boldsymbol{X}_\varepsilon$. Because of (3) this space can be expressed as

$$\boldsymbol{X}_\varepsilon^* = (\boldsymbol{L}^2 + \varepsilon^{-1}\boldsymbol{H}^{-1}) \times (\varepsilon \cdot L_0^2 \cap H^*).$$

We shall show below that the operator $\mathcal{A}_\varepsilon$ is an isomorphism mapping $\boldsymbol{X}_\varepsilon$ into $\boldsymbol{X}_\varepsilon^*$. Furthermore, the corresponding operator norms

$$||\mathcal{A}_\varepsilon||_{\mathcal{L}(X_\varepsilon, X_\varepsilon^*)} \quad \text{and} \quad ||\mathcal{A}_\varepsilon^{-1}||_{\mathcal{L}(X_\varepsilon^*, X_\varepsilon)} \quad \text{are independent of } \varepsilon. \tag{9}$$

In fact, with the definitions above, this is also true for $\varepsilon \in [0, 1]$, i.e. the endpoint $\varepsilon = 0$ can be included. However, in the discussion below we will for simplicity always assume that $\varepsilon > 0$.

The uniform boundedness of $\mathcal{A}_\varepsilon$ is straightforward to check from the definitions above. For example, if $p = p_1 + p_2$, where $p_1 \in H^1 \cap L_0^2, p_2 \in L_0^2$, then the term $(p, \operatorname{div} \boldsymbol{v})$ in (5) can be bounded by

$$\begin{aligned}
|(p, \operatorname{div} \boldsymbol{v})| &= |(p_1 + p_2, \operatorname{div} \boldsymbol{v})| \\
&= |-(\mathbf{grad}\, p_1, \boldsymbol{v}) + (p_2, \operatorname{div} \boldsymbol{v})| \\
&\leq (\|p_1\|_1 \|\boldsymbol{v}\|_0 + \|p_2\|_0 \|\operatorname{div} \boldsymbol{v}\|_0) \\
&\leq (\|p_1\|_1^2 + \varepsilon^{-2}\|p_2\|_0^2)^{1/2} |\!|\!|\boldsymbol{v}|\!|\!|_\varepsilon.
\end{aligned}$$

Hence,

$$|(p, \operatorname{div} \boldsymbol{v})| \leq |p|_\varepsilon |\!|\!|\boldsymbol{v}|\!|\!|_\varepsilon \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1, p \in L_0^2. \tag{10}$$

The uniform boundedness of $\mathcal{A}_\varepsilon^{-1}$ can be verified from the two Brezzi conditions, cf. [8]. For the present problem these conditions read:

There are constants $\alpha_0 > 0, \beta_0 > 0$, independent of ε, such that

$$\sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1} \frac{(q, \operatorname{div} \boldsymbol{v})}{|\!|\!|\boldsymbol{v}|\!|\!|_\varepsilon} \geq \alpha_0 |q|_\varepsilon \quad \forall q \in L_0^2 \tag{11}$$

and

$$a_\varepsilon(\boldsymbol{v}, \boldsymbol{v}) \geq \beta_0 |\!|\!|\boldsymbol{v}|\!|\!|_\varepsilon^2 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1. \tag{12}$$

Condition (12) obviously holds with $\beta_0 = 1$, while condition (11) will be verified in the next section.

3 Mapping properties and uniform preconditioners

As explained above the main purpose of the present paper is to construct uniform preconditioners for discrete analogs of the system (2) when this system has been discretized by standard Stokes elements. The presentation of these preconditioners will be given in the next section. However, in order to motivate these preconditioners we will in this section explain how to precondition the continuous problem. Similar discussions, where preconditioners for various discrete systems are motivated from mapping properties of the corresponding continuous systems, can for example be found in [2] and [17].

We will first establish the uniform inf–sup condition (11). If $\varepsilon = 1$ then this condition, up to equivalence of norms, reduces to

$$\sup_{\boldsymbol{v}\in \boldsymbol{H}_0^1} \frac{(q, \operatorname{div} \boldsymbol{v})}{\|\boldsymbol{v}\|_1} \geq \alpha_1 \|q\|_0 \quad \forall q \in L_0^2, \tag{13}$$

where $\alpha_1 > 0$. This is the standard inf–sup condition for the Stokes system which is well–known to hold. This result was established for a Lipschitz domain by Nečas [22], cf. also [15, Chapter 1, Corollary 2.4]. The uniform inf–sup condition (11) is now a simple consequence of (4), (13), and Poincaré's inequality.

Lemma 1 *The uniform inf–sup condition* (11) *holds.*

Proof Observe that (13) can be written in the form

$$\|\operatorname{\mathbf{grad}} q\|_{-1} \geq \alpha_1 \|q\|_0 \quad \forall q \in L_0^2.$$

Hence, if we define $\boldsymbol{G}_0 = \mathbf{grad}(L_0^2)$ then $\boldsymbol{G}_0$ is a closed subspace of $\boldsymbol{H}^{-1}$, and we can define $\mathbf{grad}^{-1} : \boldsymbol{G}_0 \mapsto L_0^2$ such that

$$\|\operatorname{\mathbf{grad}}^{-1}\|_{\mathcal{L}(\boldsymbol{G}_0, L_0^2)} \leq \alpha_1^{-1}.$$

In addition, Poincaré's inequality states that there is a constant $c = c(\Omega)$ such that

$$\|q\|_1 \leq c \|\operatorname{\mathbf{grad}} q\|_0 \quad \forall q \in H^1 \cap L_0^2,$$

or $\|\operatorname{\mathbf{grad}}^{-1}\|_{\mathcal{L}(\boldsymbol{G}_1, H^1\cap L_0^2)} \leq c$, where $\boldsymbol{G}_1 = \mathbf{grad}(H^1 \cap L_0^2)$. We therefore conclude from (4) that

$$\|\operatorname{\mathbf{grad}}^{-1}\|_{\mathcal{L}(\boldsymbol{G}_1+\varepsilon^{-1}\boldsymbol{G}_0, (H^1\cap L_0^2)+\varepsilon^{-1}L_0^2)} \leq \max(c, \alpha_1^{-1}).$$

Hence, letting $\alpha_0 = \min(\alpha_1, c^{-1})$ we obtain

$$\|\operatorname{\mathbf{grad}} q\|_{\boldsymbol{L}^2+\varepsilon^{-1}\boldsymbol{H}^{-1}} \geq \alpha_0 |q|_\varepsilon \quad \forall q \in L_0^2,$$

which is (11). □

Let $\mathcal{B}_\varepsilon : \boldsymbol{X}_\varepsilon^* \mapsto \boldsymbol{X}_\varepsilon$ be the diagonal operator

$$\mathcal{B}_\varepsilon = \begin{pmatrix} (\boldsymbol{I} - \varepsilon^2 \boldsymbol{\Delta})^{-1} & 0 \\ 0 & \varepsilon^2 I + (-\Delta)^{-1} \end{pmatrix}. \tag{14}$$

When restricted to $\boldsymbol{L}^2 \times L_0^2$ this operator is symmetric and positive definite. We observe that $\mathcal{B}_0 = \mathrm{diag}(\boldsymbol{I}, (-\Delta)^{-1})$, while $\mathcal{B}_1$ has the same mapping property as $\mathrm{diag}((-\boldsymbol{\Delta})^{-1}, I)$. In fact, it follows directly from the definitions of the spaces $\boldsymbol{X}_\varepsilon$ and $\boldsymbol{X}_\varepsilon^*$ that the operator norms

$$||\mathcal{B}_\varepsilon||_{\mathcal{L}(X_\varepsilon^*, X_\varepsilon)} \quad \text{and} \quad ||\mathcal{B}_\varepsilon^{-1}||_{\mathcal{L}(X_\varepsilon, X_\varepsilon^*)} \quad \text{are independent of } \varepsilon. \tag{15}$$

Hence, the composition

$$\mathcal{B}_\varepsilon \mathcal{A}_\varepsilon : \quad \boldsymbol{X}_\varepsilon \xrightarrow{\mathcal{A}_\varepsilon} \boldsymbol{X}_\varepsilon^* \xrightarrow{\mathcal{B}_\varepsilon} \boldsymbol{X}_\varepsilon \tag{16}$$

maps $\boldsymbol{X}_\varepsilon$ into itself. In particular, we can conclude from (9) and (15) that the operator norms

$$||\mathcal{B}_\varepsilon \mathcal{A}_\varepsilon||_{\mathcal{L}(X_\varepsilon, X_\varepsilon)}, \; ||(\mathcal{B}_\varepsilon \mathcal{A}_\varepsilon)^{-1}||_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \quad \text{are independent of } \varepsilon. \tag{17}$$

Furthermore, observe that the symmetric positive definite operator $\mathcal{B}_\varepsilon^{-1}$ defines an inner product on $\boldsymbol{X}_\varepsilon$, and that that the operator $\mathcal{B}_\varepsilon \mathcal{A}_\varepsilon$ is symmetric with respect to this inner product.

Consider now the preconditioned version of the system (5), or (7), given by

$$\mathcal{B}_\varepsilon \mathcal{A}_\varepsilon \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = \mathcal{B}_\varepsilon \begin{pmatrix} \boldsymbol{f} \\ g \end{pmatrix}, \tag{18}$$

where the operator $\mathcal{B}_\varepsilon$, introduced above, is a preconditioner. This preconditioned differential system has a symmetric and bounded coefficient operator. Therefore, the system (18) can, in theory, be solved by an iterative method like the minimum residual method (cf. for example [16]) or the conjugate gradient method applied to the normal equations. These methods are well defined as long as $\mathcal{B}_\varepsilon \mathcal{A}_\varepsilon$ maps $\boldsymbol{X}_\varepsilon$ into itself, and the convergence in the norm of $\boldsymbol{X}_\varepsilon$ can be bounded by the spectral condition number

$$\kappa(\mathcal{B}_\varepsilon \mathcal{A}_\varepsilon) = ||\mathcal{B}_\varepsilon \mathcal{A}_\varepsilon||_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \cdot ||(\mathcal{B}_\varepsilon \mathcal{A}_\varepsilon)^{-1}||_{\mathcal{L}(X_\varepsilon, X_\varepsilon)}.$$

Hence, property (17) ensures uniform convergence with respect to ε.

If the system (5) is replaced by a discrete analog, with a discrete coefficient operator $\mathcal{A}_{\varepsilon,h}$, then the corresponding preconditioner $\mathcal{B}_{\varepsilon,h}$ should also be constructed on discrete spaces. However, the continuous discussion given above suggests clearly the structure of these preconditioners. Of course, in order to obtain computational efficiency, the inverse operators appearing in the blocks of $\mathcal{B}_\varepsilon$ should then be replaced by proper elliptic preconditioners.

4 The discrete preconditioners

A standard finite element discretization of (5) leads to discrete indefinite systems approximating (5). Motivated by the continuous discussion above we will propose preconditioners for these discrete systems. The behavior of these preconditioners will then be investigated by numerical experiments, while a theoretical discussion is given in the next section.

4.1 Finite element discretization

Let $\{\boldsymbol{V}_h \times Q_h\}_{h\in(0,1]} \subset \boldsymbol{H}_0^1 \times L_0^2$ be finite element spaces, where the parameter h represents the scale of the discretization. Given the spaces $\boldsymbol{V}_h$ and Q_h the corresponding finite element discretization of the system (5) is given by:

Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that

$$\begin{aligned} a_\varepsilon(\boldsymbol{u}_h, \boldsymbol{v}) + (p_h, \operatorname{div} \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\ (\operatorname{div} \boldsymbol{u}_h, q) &= (g, q) \qquad \forall q \in Q_h. \end{aligned} \tag{19}$$

Standard stable Stokes elements satisfy a Babuska–Brezzi condition of the form

$$\sup_{\boldsymbol{v}\in\boldsymbol{V}_h} \frac{(q, \operatorname{div} \boldsymbol{v})}{\|\boldsymbol{v}\|_1} \geq \alpha_1 \|q\|_0 \quad \forall q \in Q_h, \tag{20}$$

where the positive constant α_1 is independent of the mesh parameter h. For a review of such finite element spaces we refere for example to the texts [9] and [15]. We note that (20) is a discrete version of (11) in the case when the perturbation parameter ε is bounded away from zero. This condition will imply, in particular, that the discrete system (19) has a unique solution.

The discrete system (19) can alternatively be written as a discrete analog of (7),

$$\mathcal{A}_{\varepsilon,h} \begin{pmatrix} \boldsymbol{u}_h \\ p_h \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_h \\ g_h \end{pmatrix}, \tag{21}$$

where the discrete coefficient operator $\mathcal{A}_{\varepsilon,h} : \boldsymbol{V}_h \times Q_h \mapsto \boldsymbol{V}_h \times Q_h$ is defined by

$$(\mathcal{A}_{\varepsilon,h} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix}, \begin{pmatrix} \boldsymbol{v} \\ q \end{pmatrix}) = a_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) + (p, \operatorname{div} \boldsymbol{v}) + (\operatorname{div} \boldsymbol{u}, q) \tag{22}$$

for all $(\boldsymbol{u}, p), (\boldsymbol{v}, q) \in \boldsymbol{V}_h \times Q_h$. Hence, the operator $\mathcal{A}_{\varepsilon,h}$ is an L^2–symmetric, but indefinite, operator mapping the product space $\boldsymbol{V}_h \times Q_h$ into itself.

Our goal is to construct efficient positive definite, block diagonal preconditioners for the operator $\mathcal{A}_{\varepsilon,h}$, i.e. we will construct block diagonal operators $\mathcal{B}_{\varepsilon,h} : \boldsymbol{V}_h \times Q_h \mapsto \boldsymbol{V}_h \times Q_h$ such that the condition numbers of the operators $\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h}$ are bounded uniformly in the perturbation parameter ε and the discretization parameter h. The preconditioners $\mathcal{B}_{\varepsilon,h}$, constructed below, are designed as proper discrete analogs of the operator $\mathcal{B}_\varepsilon$ introduced above.

Motivated by (14), the preconditioner $\mathcal{B}_{\varepsilon,h}$ will be constructed on the form

$$\mathcal{B}_{\varepsilon,h} = \begin{pmatrix} \boldsymbol{M}_{\varepsilon,h} & 0 \\ 0 & \varepsilon^2 I_h + N_h \end{pmatrix}. \tag{23}$$

Here $\boldsymbol{M}_{\varepsilon,h} : \boldsymbol{V}_h \mapsto \boldsymbol{V}_h$ is a preconditioner for the discrete version of the differential operator $\boldsymbol{I} - \varepsilon^2 \boldsymbol{\Delta}$ with Dirichlet boundary conditions, while the operator $N_h : Q_h \mapsto Q_h$ is a corresponding preconditioner for the discrete negative Laplacian with natural boundary conditions. Finally, the operator $I_h : Q_h \mapsto Q_h$ is the identity operator if the space Q_h consists of discontinuous functions or an operator spectrally equivalent to the identity on C^0–elements. In fact, in the present section we will only consider finite elements spaces with continuous approximations of the pressure. The reason for this is that the presence of the negative Laplacian preconditioner N_h, defined on Q_h, seems to demand that $Q_h \subset H^1$, at least as long as conforming approximations of the Laplacian are used. Hence, below we shall consider the classical Taylor–Hood element [9], [15] and the Mini element [1].

The most efficient iterative method for the preconditioned system

$$\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h} \begin{pmatrix} \boldsymbol{u}_h \\ p_h \end{pmatrix} = \mathcal{B}_{\varepsilon,h} \begin{pmatrix} \boldsymbol{f}_h \\ g_h \end{pmatrix}$$

is the preconditioned minimum residual method. We refer to cf. [16], [18], [24], [28], and [29] for general discussions of this method and block diagonal preconditioners. Alternatively, we can use the conjugate gradient method applied to the normal equations of the preconditioned system.

4.2 Numerical experiments

In order to test the behavior of the discrete preconditioners of the form (23) we will consider the system (19) with the domain Ω taken as the unit square in $\mathbb{R}^2$. A sequence of rectangular meshes is constructed by uniform refinements of a 2×2 partition of the unit square, and a triangular mesh is constructed by dividing each rectangle into two triangles by the diagonal with negative slope. The number of unknowns in the experiments below will typically range from order 10^2 to order 10^5.

In the two examples below the pressure space Q_h consists of continuous piecewise linear functions. The preconditioner $N_h : Q_h \mapsto Q_h$ is a standard V–cycle operator with a symmetric Gauss–Seidel smoother, while the approximate identity I_h on Q_h simply consists of one symmetric Gauss–Seidel iteration. The condition numbers for the operators $N_h(-\Delta_h)$ and I_h, where $\Delta_h : Q_h \mapsto Q_h$ is the corresponding discrete Laplace operator, can be estimated by a standard procedure from the preconditioned conjugate gradient method (i.e. the Lanczos algorithm), where we have chosen an oscillatory random vector as a start vector. The iteration is terminated when the residual is reduced by a factor of 10^{-17} (roughly equal to the unit round off).

A similar approach is used to estimate the condition number of $\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h}$, where we recall that $\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h}$ is symmetric with respect to the inner product generated by $\mathcal{B}_{\varepsilon,h}^{-1}$. These estimates for $\kappa(\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h})$ are based on the Conjugate Gradient method applied to the normal system

$$\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h}\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h}\begin{pmatrix}\boldsymbol{u}_h\\ p_h\end{pmatrix} = \mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h}\mathcal{B}_{\varepsilon,h}\begin{pmatrix}\boldsymbol{f}_h\\ g_h\end{pmatrix}.$$

Estimates for the condition numbers of $N_h(\Delta_h)$ and I_h are given in Table 1.

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
$\kappa(N_h(-\Delta_h))$	1.71	1.50	1.47	1.47	1.47	1.47
$\kappa(I_h)$	1.66	1.62	1.61	1.60	1.60	1.60

Table 1. Condition numbers for the operators $N_h(-\Delta_h)$ and I_h.

We observe that these operators cleary behave well as the mesh paramer h is decreased. In the examples below the preconditioners N_h and I_h are combined with proper operators $\boldsymbol{M}_{\varepsilon,h}$ to build the complete block diagonal preconditioner $\mathcal{B}_{\varepsilon,h}$ of the form (23).

Example 4.1 First we consider the preconditioner in the case of the Taylor–Hood element. Hence, $\boldsymbol{V}_h \subset \boldsymbol{H}_0^1$ consists of piecewise quadratics, while the space Q_h is the space of continuous piecewise linears. The multigrid preconditioner $\boldsymbol{M}_{\varepsilon,h} : \boldsymbol{V}_h \mapsto \boldsymbol{V}_h$, approximating $(\boldsymbol{I} - \varepsilon^2\boldsymbol{\Delta}_h)^{-1}$, where $\boldsymbol{\Delta}_h : \boldsymbol{V}_h \mapsto \boldsymbol{V}_h$ is the corresponding discrete Laplacian on $\boldsymbol{V}_h$, is a standard V–cycle operator with a symmertic Gauss–Seidel smoother. The estimates for the condition numbers

$\kappa(\boldsymbol{M}_{\varepsilon,h}(\boldsymbol{I}-\varepsilon^2\boldsymbol{\Delta}_h))$, given in Table 2, clearly indicates a bound independent of ε and h.

$h\backslash\varepsilon$	0	0.001	0.01	0.1	0.5	1.0
2^{-3}	1.11	1.11	1.03	1.14	1.22	1.22
2^{-5}	1.11	1.09	1.03	1.23	1.24	1.24
2^{-7}	1.11	1.02	1.20	1.24	1.24	1.24

Table 2. Condition numbers for $\kappa(\boldsymbol{M}_{\varepsilon,h}(\boldsymbol{I}-\varepsilon^2\boldsymbol{\Delta}_h))$ obtained from the Taylor–Hood element.

We use the operator $\boldsymbol{M}_{\varepsilon,h}$, together with the operators N_h and I_h introduced above, to construct the complete operator $\mathcal{B}_{\varepsilon,h}$ of the form (23). The estimates for the condition numbers of $\kappa(\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h})$ are given in Table 3.

$h\backslash\varepsilon$	0	0.001	0.01	0.1	0.5	1.0
2^{-3}	6.03	6.05	6.92	13.42	15.25	15.32
2^{-5}	6.07	6.23	10.62	15.14	15.59	15.61
2^{-7}	6.08	7.81	14.18	15.55	15.64	15.65

Table 3. Condition numbers for $\kappa(\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h})$ using the Taylor–Hood element.

These condition numbers appears to be independent of ε and h. This will be theoretically verified in the next section. Similar computational results as reported here are also obtained if the $P_2 - P_1$ element is replaced by the corresponding element on rectangles, i.e. the $Q_2 - Q_1$ element. □

Example 4.2 Analogous to the example with the Taylor–Hood element above, we consider the Mini element discretization, i.e. $\boldsymbol{V}_h \subset \boldsymbol{H}_0^1$ consists of piecewise linear functions and cubic bubble functions supported on a single triangle, while $Q_h \subset H^1 \cap L_0^2$ is the space of continuous piecewise linear functions. The preconditioner $\boldsymbol{M}_{\varepsilon,h} : \boldsymbol{V}_h \mapsto \boldsymbol{V}_h$, approximating $(\boldsymbol{I}-\varepsilon^2\boldsymbol{\Delta}_h)^{-1}$, is again constructed as a standard V–cycle operator with symmetric Gauss–Seidel as a smoother. However, the high frequency bubble functions are only present in the finest grid. On all the coarser grids we simply use piecewise linear functions. This approach seems to be efficient as seen by the condition number estimates in Table 4. As expected these condition numbers appears to be bounded uniformly with respect to ε and h.

$h\backslash\varepsilon$	0	0.001	0.01	0.1	0.5	1.0
2^{-3}	2.79	2.73	1.35	1.05	1.14	1.16
2^{-5}	2.94	2.22	1.02	1.15	1.20	1.21
2^{-7}	2.95	1.14	1.11	1.20	1.23	1.23

Table 4. Condition numbers for $\kappa(\boldsymbol{M}_{\varepsilon,h}(\boldsymbol{I} - \varepsilon^2 \boldsymbol{\Delta}_h))$ obtained from the Mini element.

Finally, we construct the complete operator $\mathcal{B}_{\varepsilon,h}$ of the form (23) and estimate the condition numbers of $\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h}$. The results are given in Table 5.

$h\backslash\varepsilon$	0	0.001	0.01	0.1	0.5	1.0
2^{-3}	4.32	4.23	3.59	13.87	19.18	19.43
2^{-5}	4.65	3.50	8.56	18.53	19.83	19.88
2^{-7}	4.67	4.52	15.74	19.72	19.93	19.93

Table 5. Condition numbers for $\kappa(\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h})$ obtained from the Mini element.

Again, these results seem to indicate that the condition numbers $\kappa(\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h})$ are indeed independent of ε and h. □

5 A theoretical discussion in the discrete case

The purpose of this section is to present a theoretical analysis of the preconditioners studied experimentally above. We assume that Ω is a bounded polygonal domain in $\mathbb{R}^2$ and that $\{\mathcal{T}_h\}$ is a shape regular and quasi–uniform family of triangulations of Ω, where h is the maximum diameter of a triangle in $\mathcal{T}_h$.

Below we shall give a precise analysis of the conditioning of the operator $\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h}$ when the spaces $\boldsymbol{V}_h$ and Q_h are given either by the Taylor–Hood element or the Mini element. However, first we will make some remarks in the general case. For this discussion we just assume that $\boldsymbol{V}_h \subset \boldsymbol{H}_0^1$ and $Q_h \subset H^1 \cap L_0^2$ is a pair of finite element spaces.

Let $\mathcal{A}_{\varepsilon,h} : \boldsymbol{V}_h \times Q_h \mapsto \boldsymbol{V}_h \times Q_h$ be defined by (22) and let $\mathcal{B}_{\varepsilon,h}$: $\boldsymbol{V}_h \times Q_h \mapsto \boldsymbol{V}_h \times Q_h$ be a corresponding L^2 symmetric and positive definite, block diagonal preconditioner on the form (23). Our goal is to establish bounds on the spectral condition number, $\kappa(\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h})$, which are independent of the perturbation parameter ε and the mesh parameter h. To establish this we will use the characterization

$$\kappa(\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h}) = \frac{\sup|\lambda|}{\inf|\lambda|},$$

where the supremum and infimum is taken over the spectrum of $\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h}$. The saddle point theory of [8] will be used to obtain an upper bound on $\kappa(\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h})$. Observe that if $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h}$, with corresponding eigenfunction $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$, then the equations

$$\begin{aligned} a_\varepsilon(\boldsymbol{u}_h, \boldsymbol{v}) + (p_h, \operatorname{div}\boldsymbol{v}) &= \lambda(\boldsymbol{M}_h^{-1}\boldsymbol{u}_h, \boldsymbol{v}), \\ (\operatorname{div}\boldsymbol{u}_h, q) &= \lambda((\varepsilon^2 I_h + N_h)^{-1}p_h, q), \end{aligned} \tag{24}$$

holds for all $\boldsymbol{v} \in \boldsymbol{V}_h, q \in Q_h$.

In all the examples below the operator $\boldsymbol{M}_{\varepsilon,h} : \boldsymbol{V}_h \mapsto \boldsymbol{V}_h$ will be a uniform preconditioner for the corresponding discrete version of the operator $\boldsymbol{I} - \varepsilon^2\boldsymbol{\Delta}$. In other words, the bilinear forms $a_\varepsilon(\cdot,\cdot)$ and $(\boldsymbol{M}_{\varepsilon,h}^{-1}\cdot,\cdot)$ are uniformly spectrally equivalent, i.e. there are constants c_1 and c_2, independent of ε and h, such that

$$c_1 a_\varepsilon(\boldsymbol{v},\boldsymbol{v}) \le (\boldsymbol{M}_{\varepsilon,h}^{-1}\boldsymbol{v},\boldsymbol{v}) \le c_2 a_\varepsilon(\boldsymbol{v},\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{25}$$

Furthermore, the operator $I_h : Q_h \mapsto Q_h$ will be spectrally equivalent to the identity operator, and $N_h : Q_h \mapsto Q_h$ is spectrally equivalent to the discrete Laplacian, i.e. there are constants c_3 and c_4, independent of h, such that

$$c_3\|q\|_1^2 \le (N_h^{-1}q, q) \le c_4\|q\|_1^2 \qquad \forall q \in Q_h. \tag{26}$$

As we observed in the experiments discussed in §4 above, these requirements on $\boldsymbol{M}_{\varepsilon,h}, N_h$ and I_h were easily fulfilled for the examples we studied there.

Recall that the norm $|q|_\varepsilon \equiv \|q\|_{(H^1\cap L_0^2)+\varepsilon^{-1}L_0^2}$ is characterized by

$$|q|_\varepsilon^2 = \inf_{q_1 \in H^1\cap L_0^2}[\|\operatorname{\mathbf{grad}} q_1\|_0^2 + \varepsilon^{-2}\|q - q_1\|_0^2].$$

In fact, the optimal choice is $q_1 = (I - \varepsilon^2\Delta)^{-1}q$, where Neumann boundary conditions are implicitly assumed. For functions in Q_h the corresponding discrete norm is given by

$$|q|_{\varepsilon,h} = \inf_{q_1 \in Q_h}[\|\operatorname{\mathbf{grad}} q_1\|_0^2 + \varepsilon^{-2}\|q - q_1\|_0^2].$$

It is obvious that $|q|_\varepsilon \le |q|_{\varepsilon,h}$ on Q_h. However, due to the quasi–uniformity of the triangulations $\{\mathcal{T}_h\}$, the two norms are equivalent, uniformly in h. To see this note that if $q \in Q_h$ and $q_1 = (I - \varepsilon^2 \Delta)^{-1} q$ then

$$|q|_{\varepsilon,h}^2 \le \|\operatorname{\mathbf{grad}} q_{1,h}\|_0^2 + \varepsilon^{-2}\|q - q_{1,h}\|_0^2,$$

where $q_{1,h} \in Q_h$ is the L^2 projection of q_1. However, $\|q - q_{1,h}\|_0 \le \|q - q_1\|_0$ and, by quasi-uniformity, $\|\operatorname{\mathbf{grad}} q_{1,h}\|_0 \le c_0 \|\operatorname{\mathbf{grad}} q_1\|_0$, where the constant c_0 is independent of h. Hence, we conclude that

$$|q|_{\varepsilon,h} \le c_0 |q|_\varepsilon.$$

In addition, (26) further implies that $|q|_{\varepsilon,h}$ is equivalent to the norm

$$\inf_{q_1 \in Q_h} [(N_h^{-1} q_1, q_1) + \varepsilon^{-2}\|q - q_1\|_0]^{1/2}. \qquad (27)$$

Finally, from the general properties of sums and intersections of linear spaces, cf. §2 above, it follows that the norm (27) is equivalent to the norm $((\varepsilon^2 I + N_h)^{-1} q, q)^{1/2}$.

To summarize the discussion so far we state the following result.

Lemma 2 *If $I_h, N_h : Q_h \mapsto Q_h$ are L^2 symmetric operators, such that I_h is spectrally equivalent to the identity and N_h satisfies property* (26)*, then the norms $|q|_\varepsilon$, $|q|_{\varepsilon,h}$, and $((\varepsilon^2 I_h + N_h)^{-1} q, q)^{1/2}$ are equivalent, uniformly in ε and h.*

Assume that the finite element spaces $\{\boldsymbol{V}_h \times Q_h\}$ satisfies the uniform Babuska–Brezzi condition

$$\inf_{q \in Q_h} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(q, \operatorname{div} \boldsymbol{v})}{\|\boldsymbol{v}\|_\varepsilon |q|_\varepsilon} \ge \alpha > 0, \qquad (28)$$

where α is independent of ε and h. By using the theory of [8], cf. Proposition 1.1 of that paper, this condition will imply that $\kappa(\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h})$ is bounded independently of ε and h. In fact, it is an immediate consequence of this theory, the upper bound (10), the property (25) of the operator $\boldsymbol{M}_{\varepsilon,h}$, and the norm eqivalence given in Lemma 2, that $|\lambda|$, where λ is an eigenvalue of (24), is bounded from below and above, uniformly in ε and h. We can therefore conclude with the following result.

Theorem 1 *Assume that $\boldsymbol{V}_h \subset \boldsymbol{H}_0^1$, $Q_h \subset H^1 \cap L_0^2$ and let $\boldsymbol{M}_{\varepsilon,h} : \boldsymbol{V}_h \mapsto \boldsymbol{V}_h$ be a L^2 symmetric, positive definite preconditioner satisfying* (25)*. Furthermore, assume that the operators N_h and I_h on Q_h are as in Lemma 2. If the uniform Babuska–Brezzi condition* (28) *holds then the condition numbers $\kappa(\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h})$ are bounded uniformly in ε and h.*

Hence, for any particular choice of spaces $\{\boldsymbol{V}_h \times Q_h\}$ our main task is to verify the uniform inf–sup condition (28).

5.1 The Taylor–Hood element

We recall that the velocity space, $\boldsymbol{V}_h$, consists of continuous piecewise quadratic vector fields, while the discrete pressures in Q_h are continuous and piecewise linear. The argument we will present to establish the uniform Babuska–Brezzi condition (28) resembles the continuous argument given in the proof of Lemma 1.

We first recall that it is well–known that the Taylor–Hood element satisfies (20), cf. [32] or Chapter 6 of [9]. On the other hand, for $\varepsilon = 0$ (28) takes the form

$$\inf_{q \in Q_h} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(q, \operatorname{div} \boldsymbol{v})}{\|\boldsymbol{v}\|_0 \|\,\mathbf{grad}\, q\|_0} \geq \alpha_0 > 0. \tag{29}$$

In fact, this property, which is often referred to as the weak inf–sup condition for the Taylor–Hood element, was established already in [3], at least under the restriction that no triangle has two edges on the boundary. Furthermore, this restriction can be removed by using a macro–element technique, cf. [30].

We can therefore conclude that in the two extreme cases, $\varepsilon = 0$ and $\varepsilon = 1$, the inf–sup condition is satisfied. Furthermore, these properties imply that that the weakly defined gradient, $\mathbf{grad}_h : Q_h \mapsto \boldsymbol{V}_h$ given by

$$(\boldsymbol{v}, \mathbf{grad}_h\, q) = -(\operatorname{div} \boldsymbol{v}, q) \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, q \in Q_h,$$

is one–one. Let $\boldsymbol{G}_h \subset \boldsymbol{V}_h$ be defined as $\mathbf{grad}_h(Q_h)$. As a consequence of the two estimates (20) and (29) we obtain that for all $\boldsymbol{u} \in \boldsymbol{G}_h$

$$\|\,\mathbf{grad}_h^{-1} \boldsymbol{u}\|_0 \leq \alpha_1^{-1} \|\boldsymbol{u}\|_{-1,h} \equiv \alpha_1^{-1} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{v}\|_1},$$

and

$$\|\,\mathbf{grad}_h^{-1} \boldsymbol{u}\|_1 \leq \alpha_0^{-1} \|\boldsymbol{u}\|_0.$$

From (3), (4) and Lemma 2 it therefore follows that

$$|\,\mathbf{grad}_h^{-1} \boldsymbol{u}|_\varepsilon \leq \max(\alpha_1^{-1}, \alpha_0^{-1}) \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(\boldsymbol{u}, \boldsymbol{v})}{\|\!|\boldsymbol{v}|\!\|_\varepsilon}, \quad \forall \boldsymbol{u} \in \boldsymbol{G}_h,$$

and by letting $\boldsymbol{u} = \mathbf{grad}_h\, q$ this implies (28). Hence, we have completed our theoretical explanation of the observations done in Example 4.1.

5.2 The Mini element

We recall that the velocity space, $\boldsymbol{V}_h$, consists of linear combinations of continuous piecewise linear vector fields and local cubic bubbles. More precisely, $\boldsymbol{v} \in \boldsymbol{V}_h$ if and only if

$$\boldsymbol{v} = \boldsymbol{v}^1 + \sum_{T \in \mathcal{T}_h} \boldsymbol{c}_T b_T,$$

where $\boldsymbol{v}^1$ is a continuous piecewise linear vector field, $\boldsymbol{c}_T \in \mathbb{R}^2$, and b_T is the scalar cubic bubble function with respect to T, i.e. the unique cubic function vanishing on ∂T and with $\int_T b_T \, dx$ equals the area of T. The pressure space Q_h is the standard space of continuous piecewise linear scalar fields.

As above we can establish the uniform Babuska-Brezzi condition provided that the extreme cases $\varepsilon = 0$ and $\varepsilon = 1$ are valid. The case $\varepsilon = 1$ is well–known, cf. [1]. It remains to show the weak inf-sup condition (29). However, in the present case a direct argument for (29) is straightforward.

First observe that the continuous analog of (29) obviously holds with $\alpha_0 = 1$. Therefore, it is enough to construct and an interpolation operator $\boldsymbol{\Pi}_h : \boldsymbol{L}^2 \mapsto \boldsymbol{V}_h$, uniformly bounded with respect to h, such that such that

$$(\operatorname{div} \boldsymbol{\Pi}_h \boldsymbol{v}, q) = (\operatorname{div} \boldsymbol{v}, q) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1, q \in Q_h. \tag{30}$$

Define $\boldsymbol{\Pi}_h : \boldsymbol{L}^2 \mapsto \boldsymbol{V}_h^b \subset \boldsymbol{V}_h$ by

$$(\boldsymbol{\Pi}_h \boldsymbol{v}, \boldsymbol{z}) = (\boldsymbol{v}, \boldsymbol{z}) \qquad \forall \boldsymbol{z} \in \boldsymbol{Z}_h,$$

where $\boldsymbol{Z}_h$ denotes the space of piecewise constants vector fields, and $\boldsymbol{V}_h^b$ denotes the span of the bubble functions. Clearly this uniquely determines $\boldsymbol{\Pi}_h$, and a scaling argument, utilizing equivalence of norms, shows that the local operators $\boldsymbol{\Pi}_h$ are uniformly bounded, with respect to h in L^2. Furthermore, property (30) follows since for all $\boldsymbol{v} \in \boldsymbol{H}_0^1$ and $q \in Q_h$, we have

$$(\operatorname{div} \boldsymbol{\Pi}_h \boldsymbol{v}, q) = -(\boldsymbol{\Pi}_h \boldsymbol{v}, \mathbf{grad}\, q) = -(\boldsymbol{v}, \mathbf{grad}\, q) = (\operatorname{div} \boldsymbol{v}, q),$$

where we have used that $\mathbf{grad}\, Q_h \subset \boldsymbol{Z}_h$. The uniform inf–sup condition (28) has therefore been established in this case.

6 Discontinuous approximation of the pressure

In the analysis above we have strongly utilized the fact that we have continuous discrete pressures, i.e. the pressure space Q_h is a subspace of H^1. In fact, the bilinear form associated with the preconditioner $N_h : Q_h \mapsto Q_h$ is required to be equivalent to the H^1 inner product on Q_h. However, several common Stokes elements use discontinuous piecewise constant pressures. The construction of uniform preconditioners for the coefficient operator $\mathcal{A}_{\varepsilon,h}$ in these cases will be discussed in this section. For simplicity, we restrict the discussion to the well known $P_2 - P_0$ element, but we have also seen similar behavior as we will present below in numerical experiments with other elements like the nonconforming Crouzeix-Raviart element [11].

Let $Q_h \subset L_0^2$ be the space of discontinuous constant functions with respect to a triangulation $\mathcal{T}_h$, where, as above, $\{\mathcal{T}_h\}$ is a shape regular and quasi–uniform family of triangulations of Ω. As for the Taylor–Hood element discussed above, the velocity space $\boldsymbol{V}_h \subset \boldsymbol{H}_0^1$ consists of continuous, piecewise quadratic vector fields. A weakly defined gradient operator $\mathbf{grad}_h : Q_h \mapsto \boldsymbol{V}_h$ is given by

$$(\mathbf{grad}_h\, q, \boldsymbol{v}) = -(q, \operatorname{div} \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h,\ q \in Q_h.$$

A discrete analog of the norm on $(H^1 \cap L_0^2) + \varepsilon^{-1} \cdot L_0^2$ can now be defined on Q_h as

$$|q|_{\varepsilon,h} = \inf_{\substack{q=q_1+q_2 \\ q_1,q_2 \in Q_h}} (\|\,\mathbf{grad}_h\, q_1\|_0^2 + \varepsilon^{-2}\|q_2\|_0^2)^{1/2}.$$

The appropriate uniform inf–sup condition we are seeking takes the form

$$\inf_{q \in Q_h} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(q, \operatorname{div} \boldsymbol{v})}{\|\boldsymbol{v}\|_\varepsilon |q|_{\varepsilon,h}} \geq \alpha > 0, \tag{31}$$

for a suitable α independent of ε and h.

For the $P_2 - P_0$ element the standard inf–sup condition (20) is well–known, cf. for example [9, Chapter 6.4]. In particular, this implies that $\mathbf{grad}_h : Q_h \mapsto \boldsymbol{V}_h$ is one–one. Furthermore, (20) implies the discrete Poincaré inequality

$$\|q\|_0 \leq \alpha_1^{-1} \|\,\mathbf{grad}_h\, q\|_0 \quad \forall q \in Q_h. \tag{32}$$

It is also straightforward to check that the norms $\|q\|_0$ and $|q|_{1,h}$ are equivalent on Q_h, uniformly in h. To see this note that $|q|_{1,h} \leq \|q\|_0$ is a direct consequence of the definition of $|q|_{1,h}$. On the other hand, if q_1 is chosen as the minimizer in the definition of $|q|_{1,h}$ we have

$$|q|_{1,h}^2 = \|\,\mathbf{grad}_h\, q_1\|_0^2 + \|q - q_1\|_0^2$$

and

$$(\mathbf{grad}_h\, q_1, \mathbf{grad}_h\, r) + (q_1, r) = (q, r) \quad \forall r \in Q_h.$$

From (32) we then obtain

$$\begin{aligned} \|q\|_0^2 &= (q, q - q_1) + (q, q_1) \\ &= (q, q - q_1) + \|q_1\|_0^2 + \|\,\mathbf{grad}_h\, q_1\|_0^2 \\ &\leq \frac{1}{2}\|q\|_0^2 + (1 + \alpha_1^{-2})|q|_{1,h}^2. \end{aligned}$$

Therefore, $\|q\|_0$ and $|q|_{1,h}$ are uniformly equivalent on Q_h, and (31) for $\varepsilon = 1$ follows from (20).

When $\varepsilon = 0$ (31) holds with constant $\alpha = 1$. This is a direct consequence of the definitions of $\mathbf{grad}_h$ and $|\cdot|_{\varepsilon,h}$. The uniform inf-sup condition (31) therefore follows for all $\varepsilon \in [0, 1]$ by an argument completely analog to the one given in §5.1 above.

Having established a uniform inf–sup condition we are again in position to construct a uniform, block diagonal preconditioner for the operator $\mathcal{A}_{\varepsilon,h}$ using similar arguments as above. Consider an operator of the form

$$\mathcal{B}_{\varepsilon,h} = \begin{pmatrix} \boldsymbol{M}_{\varepsilon,h} & 0 \\ 0 & \varepsilon^2 I_h + N_h \end{pmatrix} \tag{33}$$

mapping $\boldsymbol{V}_h \times Q_h$ into itself. In fact, in the present case, where the pressure space Q_h is discontinuous, we simply take I_h to be the identity operator. Furthermore, as in §5.2 above we have to our disposal a uniform preconditioner $\boldsymbol{M}_{\varepsilon,h} : \boldsymbol{V}_h \mapsto \boldsymbol{V}_h$ for the discrete version of the differential operator $\boldsymbol{I} - \varepsilon^2 \boldsymbol{\Delta}$ on the piecewise quadratic space $\boldsymbol{V}_h$.

It only remains to specify the symmetric and positive definite preconditioner $N_h : Q_h \mapsto Q_h$. Assume that we can construct N_h such that the norms $\|\,\mathbf{grad}_h\, q\|_0$ and $(N_h^{-1} q, q)^{1/2}$ are equivalent, uniformly in h, on Q_h. As in §5 above it then follows from the uniform inf–sup condition (31), some obvious upper bounds, and the theory of [8] that the condition number $\kappa(\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h})$ is bounded independently of ε and h.

A potential difficulty for the construction of the preconditioner N_h is that the operator $\mathbf{grad}_h : Q_h \mapsto \boldsymbol{V}_h$ is nonlocal. However, there is a local norm, $\|q\|_{1,h}$, which is equivalent to $\|\,\mathbf{grad}_h\, q\|_0$, and the structure of this local norm can more easily be used to construct the preconditioner N_h. Define a new norm on Q_h by

$$\|q\|_{1,h}^2 = \sum_{e \in \mathcal{E}_h} [q]_e^2, \tag{34}$$

where $\mathcal{E}_h$ is the set of interior edges of $\mathcal{T}_h$ and $[q]_e$ is the jump of q on the edge e.

Lemma 3 *The norms* $\|\mathbf{grad}_h\, q\|_0$ *and* $\|q\|_{1,h}$ *are equivalent on* Q_h*, uniformly in* h*.*

Proof The standard degrees of freedom for the space $\boldsymbol{V}_h$ is the function values at each vertex and the zero order moments on each edge. As a consequence of equivalence of norms we therefore obtain, from a standard scaling argument, that

$$\sum_{e\in\mathcal{E}_h}(\int_e \boldsymbol{v}\cdot\boldsymbol{n}_e\,d\rho)^2 \le c\|\boldsymbol{v}\|_0^2 \qquad \forall \boldsymbol{v}\in\boldsymbol{V}_h,$$

where c is a constant independent of h. Here $\boldsymbol{n}_e$ is a unit normal vector on the edge e and ρ is the arc length along e. As a consequence, for any $\boldsymbol{v}\in\boldsymbol{V}_h$ and $q\in Q_h$ we have

$$\begin{aligned}(\mathbf{grad}_h\, q,\boldsymbol{v}) &= -\sum_{T\in\mathcal{T}_h}\int_T q\operatorname{div}\boldsymbol{v}\,dx = -\sum_{e\in\mathcal{E}_h}[q]_e\int_e \boldsymbol{v}\cdot\boldsymbol{n}_e\,d\rho\\ &\le \|q\|_{1,h}(\sum_{e\in\mathcal{E}_h}(\int_e \boldsymbol{v}\cdot\boldsymbol{n}_e\,d\rho)^2)^{1/2} \le c\|q\|_{1,h}\|\boldsymbol{v}\|_0,\end{aligned}$$

and we can therefore conclude that

$$\|\mathbf{grad}_h\, q\|_0 \le c\|q\|_{1,h} \quad \forall q\in Q_h.$$

To establish the opposite bound let $q\in Q_h$ be given and define $\hat{\boldsymbol{v}}\in\boldsymbol{V}_h$ such that $\hat{\boldsymbol{v}}$ is zero at each vertex, the tangential component of the zero order moments are zero on each edge, and

$$\int_e \hat{\boldsymbol{v}}\cdot\boldsymbol{n}_e\,d\rho = -[q]_e$$

for all $e\in\mathcal{E}_h$. Again, equivalence of norms implies that

$$\|\hat{\boldsymbol{v}}\|_0^2 \le c\sum_{e\in\mathcal{E}_h}(\int_e \hat{\boldsymbol{v}}\cdot\boldsymbol{n}_e\,d\rho)^2 = c\sum_{e\in\mathcal{E}_h}[q]_e^2,$$

where the constant c is independent of h. Hence,

$$\begin{aligned}(\mathbf{grad}_h\, q,\hat{\boldsymbol{v}}) = -\sum_{e\in\mathcal{E}_h}[q]_e\int_e \hat{\boldsymbol{v}}\cdot\boldsymbol{n}_e\,d\rho &= \sum_{e\in\mathcal{E}_h}[q]_e^2\\ &\ge c^{-1}\|q\|_{1,h}\|\hat{\boldsymbol{v}}\|_0,\end{aligned}$$

which implies that

$$\|q\|_{1,h} \le c\|\mathbf{grad}_h\, q\|_0.$$

This completes the proof. □

6.1 Numerical experiments

Our purpose is to repeat the experiments we did in the Examples 4.1 and 4.2, but this time we use the $P_2 - P_0$ element for the discretization. In order to complete the description of the preconditioner $\mathcal{B}_{\varepsilon,h}$ given by (33) we have to make a proper choice for the preconditioner N_h on Q_h. However, due to Lemma 3 the operator N_h can be constructed as any preconditioner for the "finite difference Laplacian" obtain from the bilinear form associated the norm $\| \cdot \|_{1,h}$, cf. (34). This can be done in many ways, cf. for example [31, Chapter 3] or [27]. Due to implementational convenience we shall here adopt the auxiliary space technique of Xu [35], where the auxiliary space consists of piecewise linear functions. The advantage with this approach is that N_h is essentially constructed from the corresponding preconditioner introduced in §4.2 above. In the Examples 4.1 and 4.2, the subspace of $H^1 \cap L^2_0$ consisting of continuous piecewise linear functions with respect to the triangulation $\mathcal{T}_h$ was denoted Q_h, but here, where Q_h already denotes the space of discontinuous constants, we will refer to this space as S_h.

Let $P_h : S_h \mapsto Q_h$ be the L^2 projection, and $P_h^* : Q_h \mapsto S_h$ the adjoint operator with respect to the L^2 inner product. The preconditioner N_h we shall use will be of the form

$$N_h = \tau h^2 I + P_h N_h^S P_h^*, \tag{35}$$

where $N_h^S : S_h \mapsto S_h$ is the standard V–cycle multigrid preconditioner for the discrete Laplacian on S_h, described in §4.2 above, while $\tau > 0$ is a suitable scaling constant. In the experiments below $\tau = 0.15$. The preconditioner N_h is computationally feasible since the L^2 projection P_h is local.

First, we check the efficiency of the preconditioner N_h by computing the condition numbers of $N_h(-\Delta_h)$, where $\Delta_h : Q_h \mapsto Q_h$ is given by

$$(-\Delta_h p, q) = \sum_{e \in \mathcal{E}_h} [p]_e [q]_e \quad \forall p, q \in Q_h.$$

The results, which are given in Table 6, clearly indicate that $\kappa(N_h(-\Delta_h))$ is bounded independently of h. In fact, a theoretical verification of this can be done following the theory outlined in [35]. More precisely, it can be shown that

$$\|q\|_{1,h} \equiv (-\Delta_h q, q)^{1/2} \quad \text{and} \quad (N_h \Delta_h q, \Delta_h q)^{1/2} \tag{36}$$

are equivalent on Q_h. However, since any oparator N_h satisfing (36) is suitable, we will not include the proof here.

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
$\kappa(N_h(-\Delta_h))$	3.07	3.13	3.16	3.18	3.17	3.18

Table 6. Condition numbers for the operators $N_h(-\Delta_h)$.

Having verified, at least experimentally, that N_h is a uniform preconditioner for $-\Delta_h$ we should expect that the operator $\mathcal{B}_{\varepsilon,h}$, given by (33), is a uniform preconditioner for the operator $\mathcal{A}_{\varepsilon,h}$.

The observed condition numbers of $\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h}$, for $\varepsilon \in [0,1]$ and decreasing values of h, are given in Table 7. In complete agreement with the prediction of the theory these condition numbers appears to be bounded independently of ε and h.

$h\backslash\varepsilon$	0	0.001	0.01	0.1	0.5	1.0
2^{-3}	4.96	4.95	4.48	5.56	7.85	7.79
2^{-5}	5.22	5.07	4.46	7.12	8.72	8.74
2^{-7}	5.28	4.30	5.93	8.27	9.24	9.28

Table 7. Condition numbers for $\kappa(\mathcal{B}_{\varepsilon,h}\mathcal{A}_{\varepsilon,h})$ using the $P_2 - P_0$ element.

References

1. D.N. Arnold, F. Brezzi and M. Fortin, A stable finite element method for the Stokes equations, Calcolo 21 (1984), pp. 337–344.
2. D.N. Arnold, R.S. Falk and R. Winther, Preconditioning discrete approximations of the Reissner–Mindlin plate model, M^2AN 31 (1997), pp 517–557.
3. M. Bercovier and O. Pironneau, Error estimates for finite element method solution of the Stokes problem in primitive variables, Numer. Math. 33 (1979), pp. 211–224.
4. J. Bergh and J. Löfström, *Interpolation spaces*, Springer Verlag, 1976.
5. D. Braess and C. Blömer, A multigrid method for a parameter dependent problem in solid mechanics, Numer. Math 57 (1990), pp. 747–761.
6. J.H. Bramble and J.E. Pasciak, Iterative techniques for time dependent Stokes problem, Comput. Math. Appl. 33 (1997), pp. 13–30.
7. S. C. Brenner, Multigrid methods for parameter dependent problems, RAIRO M^2AN, 30 (3) (1996), pp. 265–297.
8. F. Brezzi, On the existence, uniqueness and approximation of saddle–point problems arising from Lagrangian multipliers, RAIRO Anal. Numér. 8 (1974), pp. 129–151.
9. F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer Verlag, 1991.
10. J. Cahouet and J. P. Chabard, Some fast 3D finite element solvers for the generalized Stokes problem. Int. Journal for Numerical Methods in Fluids, 8 (1988), pp. 869–895.

11. M. Crouzeix and P.A. Raviart, Conforming and non–conforming finite element methods for solving the stationary Stokes equations, RAIRO Anal. Numér. 7 (1973), pp. 33–76.
12. E.J. Dean and R. Glowinski, *On Some Finite Element Methods for the Numerical Simulation of Incompressible Viscous Flow*, In Proceedings: Incompressible computational fluid dynamics; Trends and advances, Editors: M. D. Gunzburger and R. A. Nicolaides, Cambridge University Press, 1993.
13. H. C. Elman, Preconditioners for saddle point problems arising in computational fluid dynamics, Appl. Numer. Math. 43 (2002), pp. 75–89.
14. H. C. Elman, D. J. Silvester and A. J. Wathen, Block preconditioners for the discrete incompressible Navier–Stokes equations, Int. J. Numer. Meth. Fluids, 40 (2002), pp. 333-344.
15. V. Girault and P.-A. Raviart, *Finite element methods for Navier–Stokes equations*, Springer Verlag 1986.
16. W. Hackbusch, *Iterative solution of large sparse systems of equations*, Springer Verlag 1994.
17. E. Haug and R. Winther, A domain embedding preconditioner for the Lagrange multiplier system, Math. Comp. 69 (1999), pp. 65–82.
18. A. Klawonn, An optimal preconditioner for a class of saddle point problems with a penalty term, SIAM J. Sci. Comput. 19 (1998), pp. 540–552.
19. D. Loghin and A. J. Wathen, Schur complement preconditioners for the Navier-Stokes equations, Int. J. Numer. Meth. Fluids, 40 (2002), pp. 403–412.
20. Y. Mayday, D. Meiron, A. Patera, E. Ronquist, Analysis of iterative methods for the steady and unsteady Stokes problem: application to spectral element discretizations, SIAM J. Sci. Comput. 14 (1993), pp. 310–337.
21. K.A. Mardal, X.-C. Tai and R. Winther, A robust finite element method for Darcy–Stokes flow, SIAM J. Numer. Anal. 40 (2002), pp. 1605–1631.
22. J. Nečas, *Equations aux dérivée partielles*, Presses de l'Université de Montréal 1965.
23. M.A. Olshanskii and A. Reusken, Navier–Stokes equations in rotation form: A robust multigrid solver for the velocity problem, SIAM J. Sci. Comp. 23 (2002), pp. 1683–1706.
24. C.C Paige and M.A. Sauders, Solution of sparse indefinite systems of linear equations, SIAM J. Numer. Anal. 12 (1975), pp. 617–629.
25. L. F. Pavarino, Indefinite overlapping Schwarz methods for time-dependent Stokes problems, Comput. Meth. Appl. Mech. Eng. 187 (2000), 35–51.
26. O. Pironneau, *The finite element method for fluids*, John Wiley & Sons, 1989.
27. T. Rusten, P.S. Vassilevski and R. Winther, Interior penalty preconditioners for mixed finite element approximations of elliptic problems, Math. Comp. 65 (1996), pp. 447–466.
28. T. Rusten and R. Winther, A preconditioned iterative method for saddle point problems, SIAM J. Matrix Anal. 13 (1992), pp. 887–904.
29. D. Silvester and A. Wathen, Fast iterative solution of stabilized Stokes systems. Part II: Using block diagonal preconditioners, SIAM J. Numer. Anal. 31 (1994), pp. 1352–1367.
30. R. Stenberg, Error analysis of some finite element methods for the Stokes problem, Math. Comp. 54 (1990), pp. 495–508.
31. S. Turek, *Efficient Solvers for Incompressible Flow Problem*, Springer Verlag 1999.

32. R. Verfürth, Error estimates for a mixed finite element approximation of the Stokes equation, R.A.I.R.O. Anal. Numer. 18 (1984), pp. 175–182.
33. R. Verfürth, A multilevel algorithm for mixed problems, SIAM J. Numer. Anal, 21 (1984), pp. 264–271.
34. G. Wittum, Multigrid methods for Stokes and Navier-Stokes equations, Numer. Math. 54 (1989) pp. 543-564.
35. J. Xu, The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grids, Computing 56, (1996) pp. 215–235.