Int J Geomath (2014) 5:81–98
DOI 10.1007/s13137-013-0054-9

ORIGINAL PAPER

Regularized collocation for spherical harmonics gravitational field modeling

Valeriya Naumova · Sergei V. Pereverzyev · Pavlo Tkachenko

Received: 5 September 2013 / Accepted: 30 October 2013 / Published online: 17 November 2013

Abstract Motivated by the problem of satellite gravity gradiometry, which is the reconstruction of the Earth gravity potential from the satellite data provided in the form of the second-order partial derivatives of the gravity potential at a satellite altitude, we discuss a special regularization technique for solving this severely ill-posed problem in a spherical framework. We are especially interested in the regularized collocation method. As a core ingredient we present an a posteriori parameter choice rule, namely the weighted discrepancy principle, and prove its order optimality. Finally, we illustrate our theoretical findings by numerical results for the computation of the Fourier coefficients of the gravitational potential directly from the noisy synthetic data.

Keywords Ill-posed problem · Collocation method · Regularization · Discrepancy principle · Satellite gravity modeling · Spherical harmonics

Mathematics Subject Classification (2000) 65J20 · 47A52 · 86A30

1 Introduction

Satellite missions, Gravity recovery and climate experiment (GRACE) [see, e.g., Tapley et al. (2005)] and Gravity field and steady-state Ocean Circulation Explorer (GOCE) [see, e.g., Rebhan et al. (2000)] launched in 2005 and 2009 respectively, are dedicated to measuring the Earth's gravity field and modeling the geoid that allow us

Dedicated to Willi Freeden's 65th Birthday.

V. Naumova · S. V. Pereverzyev · P. Tkachenko (✉)
Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenbergerstrasse 69, 4040 Linz, Austria
e-mail: pavlo.tkachenko@oeaw.ac.at

to increase our knowledge and reveal many fascinating things in studying dynamic processes in the Earth's interior, ocean circulation, etc. After collecting data from a satellite orbit the following problem naturally arises: "How to transform the satellite data into parameters of the gravitational field model?"

At this point it is worth to mention that, on the one hand, in the existing models, such as Earth Gravity Model (EGM2008) (Pavlis et al. 2008), for example, the gravitational potential is parametrized by the Fourier coefficients with respect to the spherical harmonics up to some degree M. On the other hand, the satellite data collected during a mission such as GOCE are given as the values of the second-order partial derivatives of the gravitational potential calculated at the satellite orbit. Of particular interest, from the mathematical point of view, is the use of the second-order radial derivatives, which indeed can be found from the above mentioned values.

In the spherical framework, using the second-order radial derivatives on the orbital sphere Ω_ρ, one can relate the satellite data and the parameters of the gravity model by means of the so-called gravity gradiometry equation with the operator $A : L^2(\Omega_R) \to L^2(\Omega_\rho)$

$$Ax(t) := \int\limits_{\Omega_R} h(t,\tau)x(\tau)d\Omega_R(\tau) = y(t), \ t \in \Omega_\rho, \tag{1}$$

where $h(t,\tau) = \frac{1}{4\pi R}\frac{\partial^2}{\partial\rho^2}\left[\frac{\rho^2-R^2}{(\rho^2+R^2-2t\tau)^{3/2}}\right]$, R is the radius of the Earth, ρ is the radius of the satellite orbit, Ω_R denotes the surface of the Earth, x is the gravitational potential considered at the reference sphere Ω_R, and y is its second-order radial derivative derived from the satellite data.

Moreover, a straightforward calculation (Freeden 1999) shows that the operator A admits the singular value decomposition

$$A = \sum_{i=0}^{\infty} a_i u_i \left\langle v_i, \cdot \right\rangle_{L^2(\Omega_R)}, \tag{2}$$

where

$$u_i = u_i(t) = \frac{1}{\rho} Y_{k,j}\left(\frac{t}{\rho}\right), \ v_i = v_i(\tau) = \frac{1}{R} Y_{k,j}\left(\frac{\tau}{R}\right),$$
$$i = j + k^2, \ j = 1, 2, \ldots, 2k+1, \ k = 0, 1, \ldots,$$

here $\langle\cdot,\cdot\rangle_{L^2(\Omega_R)}$ is the standard inner product in the Hilbert space $L^2(\Omega_R)$, $\left\{Y_{k,j}\right\}$ is a system of spherical harmonics orthonormalized with respect to $\langle\cdot,\cdot\rangle_{L^2(\Omega_1)}$ on the unit sphere Ω_1, and

$$a_i = \left(\frac{R}{\rho}\right)^k \frac{(k+1)(k+2)}{\rho^2}, \tag{3}$$
$$i = j + k^2, j = 1, 2, \ldots, 2k+1, \ \ k = 0, 1, \ldots,$$

are the singular values of the operator A.

In view of the decomposition (2) the solution of the Eq. (1) has the form

$$x(\tau) = \sum_{k=0}^{\infty} \sum_{j=1}^{2k+1} \hat{x}_{k,j} \frac{1}{R} Y_{k,j}\left(\frac{\tau}{R}\right),$$

where

$$\hat{x}_{k,j} = \frac{1}{R} \int\limits_{\Omega_R} x(\tau) Y_{k,j}\left(\frac{\tau}{R}\right) d\Omega_R(\tau) \tag{4}$$

are the Fourier coefficients of the gravitational potential with respect to the spherical harmonics.

At this point it should be obvious that the gravity gradiometry Eq. (1) relates the satellite data y with the parameters of the gravity model, which as mentioned above, are the Fourier coefficients.

In practice we are given just finite amount of points $\{\tau_i\}_{i=1}^n \subset \Omega_\rho$ at which our satellite data are provided. We will call these points as the collocation points or, simply, nodes. It should be noted further that due to a measurement error the data $\{y(\tau_i)\}_{i=1}^n$ are not available. Instead we are provided only with noisy measurements $y^n = (y_1, y_2, \ldots, y_n)$ such that

$$|y(\tau_i) - y_i| \le \epsilon_i, \quad i = 1, 2, \ldots, n, \tag{5}$$

where $\epsilon_i,\ i = 1, 2, \ldots, n$, are data errors.

Thus, it is natural to consider a discretized and noisy version of the Eq. (1), that can be formally written

$$T_n A x = y^n, \tag{6}$$

where the operator $T_n : L^2(\Omega_\rho) \to \mathbb{R}^n$ is such that

$$T_n f = (f(\tau_1), f(\tau_2), \ldots, f(\tau_n)),\ \forall f \in L^2(\Omega_\rho).$$

It is also worth to recall that the existing Earth gravity models are parametrized by the finite number M of the Fourier coefficients (4), such that the Eq. (6) can be discretized even further, namely we get

$$T_n A Q_M x = y^n, \tag{7}$$

where

$$Q_M = \sum_{k=0}^{M} \sum_{j=1}^{2k+1} \frac{1}{R} Y_{k,j}\left(\frac{\cdot}{R}\right) \left\langle \frac{1}{R} Y_{k,j}\left(\frac{\cdot}{R}\right), \cdot \right\rangle_{L^2(\Omega_R)}$$

is the orthoprojector onto the finite-dimensional space spanned by the spherical harmonics up to degree M, $\text{rank}(Q_M) = (M + 1)^2$.

At this point it is important to note that due to the nature of downward continuation the original problem (1) is known to be severely ill-posed (Freeden and Pereverzev 2001; Bauer et al. 2007), and this ill-posedness is inherited by the Eq. (7) in the form of the ill-conditioning of the corresponding linear system. Therefore, a regularization technique should be employed for solving (7) (Engl et al. 1996).

Note also that the Eq. (7) can be seen as a result of a combination of the collocation and the projection methods applied to the Eq. (1). The combination of these methods was studied in Le Gia and Mhaskar (2006). However, in that article no regularization was considered since the data y were assumed to be noise-free. Moreover, in contrast to (3) the singular values of the corresponding operators in Le Gia and Mhaskar (2006) were not assumed to decay exponentially.

In regularization theory the collocation method and the projection scheme have been studied so far only separately. As to the collocation, it has been studied in Nair and Pereverzev (2007), but the analysis considered there cannot be directly applied to our problem since it corresponds to the case $M = \infty$ and the solution of the problem (1) is provided in the form of the finite sum $\sum_{i=1}^{n} c_i h(\tau_i, \cdot)$ that does not correspond to any of the existing Earth gravity models.

As to the projection scheme, it has been studied extensively and we refer the reader, for example, to Plato and Vainikko (1990); Mathé and Pereverzev (2006b) for more details. However, in this method the orthoprojectors are applied to both sides of the initial Eq. (1). Therefore, if we use this scheme directly in our case, we have to be given the right-hand side of (1) in the form of the Fourier coefficients that requires post-processing of the satellite data.

To the best of our knowledge, regularization of the Eq. (7) has not appeared in the literature so far and, thus, theoretical and numerical investigations are needed.

The paper is organized as follows: in the next section we present a relation between the collocation and the projection methods. In Sect. 3 we present a convergence analysis of the regularized collocation for our problem. Section 4 is devoted to the analysis of an a posteriori parameter choice rule, namely we present the weighted discrepancy principle and show its order-optimality. Finally, in the last section we present some numerical experiments confirming the theoretical results from previous sections.

2 A bridge between collocation and projection

We start with the observation that in view of (2) the Eq. (7) can be rewritten as

$$T_n P_M A x = y^n, \tag{8}$$

where

$$P_M = \sum_{k=0}^{M} \sum_{j=1}^{2k+1} \frac{1}{\rho^2} Y_{k,j}\left(\frac{\cdot}{\rho}\right) \left\langle Y_{k,j}\left(\frac{\cdot}{\rho}\right), \cdot \right\rangle_{L^2(\Omega_\rho)} \tag{9}$$

is an orthoprojector in $L^2(\Omega_\rho)$.

It is clear that the Eqs. (7), (8) should be considered in n-dimensional vector space $\mathbb{R}^n$, where many inner products and corresponding norms can be introduced. In general, such inner products are defined as

$$\langle u, v\rangle_\omega := \sum_{i=1}^{n} \omega_i^n u_i v_i, \quad u, v \in \mathbb{R}^n,$$

where $\omega = (\omega_1^n, \omega_2^n, \ldots, \omega_n^n) \in \mathbb{R}^n$ is a vector with positive components. Let us denote by $\mathbb{R}_\omega^n$ the vector space $\mathbb{R}^n$ that is equipped with the inner product $\langle\cdot,\cdot\rangle_\omega$ and the corresponding norm $\|u\|_\omega := \langle u, u\rangle_\omega^{1/2}$, $u \in \mathbb{R}^n$.

As it has been already mentioned, the Eqs. (7), (8) are, in general, ill-conditioned, and should be treated using a regularization method. Tikhonov-Phillips regularization (Tikhonov 1963) is one of the most widely used methods for solving ill-posed problems. In this case, an approximate solution of the Eqs. (7), (8) is defined as the minimizer of the functional

$$J_\alpha(x) = \left\| B_{n,M}x - y^n \right\|_\omega^2 + \alpha \, \|x\|_{L^2(\Omega_R)}^2,$$

where $B_{n,M} = T_n P_M A = T_n A Q_M : L^2(\Omega_R) \to \mathbb{R}_\omega^n$ and α is the regularization parameter.

The minimizer $x = x_{\alpha,\delta}$ of the functional $J_\alpha(x)$ can be written in the form

$$x_{\alpha,\delta} = x_{\alpha,\delta}(B_{n,M}) = (\alpha I + B_{n,M}^* B_{n,M})^{-1} B_{n,M}^* y^n, \tag{10}$$

where

$$B_{n,M}^* u = \sum_{k=0}^{M} \sum_{j=1}^{2k+1} \frac{1}{\rho^2} Y_{k,j}\left(\frac{t}{\rho}\right) \sum_{l=1}^{n} \omega_l^n Y_{k,j}\left(\frac{\tau_l}{\rho}\right) u_l$$

is the adjoint of the operator $B_{n,M} : L^2(\Omega_R) \to \mathbb{R}_\omega^n$.

It is well-known [see, e.g., Groetsch (1990); Rajan (2003); Mathé and Pereverzev (2006b)] that if regularization of the equation $Ax = y$ is carried out by an approximation of the form $(\alpha I + B^*B)^{-1}B^*y$ then the quantity $\|A^*A - B^*B\|$ plays a crucial role, and one is interested in having it as small as possible. On the other hand, if $\operatorname{rank}(A) = \infty$ and $\operatorname{rank}(B) \le \nu$ then from Pietsch (1980) it is known that $\|A^*A - B^*B\| \ge a_{\nu+1}^2(A)$, where $a_{\nu+1}(A)$ is the $(\nu + 1)$-th singular value of the operator A.

In our context, the operator A is given by the Eqs. (1) and (2), $B = B_{n,M}$ with $\operatorname{rank}(B_{n,M}^* B_{n,M}) \le (M + 1)^2$, and the question is whether it is possible to choose a vector of weights $\omega = (\omega_1^n, \omega_2^n, \ldots, \omega_n^n)$ such that

$$\left\| A^*A - B_{n,M}^* B_{n,M} \right\|_{L^2(\Omega_R)\to L^2(\Omega_R)} = a_{(M+1)^2+1}^2, \tag{11}$$

where $a_{(M+1)^2+1}$ is defined by (3) with $i=(M+1)^2+1$. The following lemma gives a positive answer to this question.

Lemma 2.1 *If the vector of positive weights $\omega=(\omega_1^n,\omega_2^n,\dots,\omega_n^n)$ is chosen in such a way that a cubature formula*

$$\int_{\Omega_\rho} f(\varsigma)d\Omega_\rho(\varsigma)\approx\sum_{i=1}^n \omega_i^n f(\tau_i) \tag{12}$$

is exact for all spherical polynomials f of 3 variables up to degree $2M$, then

$$\left\|A^*A-B^*_{n,M}B_{n,M}\right\|_{L^2(\Omega_R)\to L^2(\Omega_R)}=a^2_{(M+1)^2+1}. \tag{13}$$

Proof From (2) and (9) it follows that

$$\left\|A^*A-A^*P_MA\right\|_{L^2(\Omega_R)\to L^2(\Omega_R)}=a^2_{(M+1)^2+1}. \tag{14}$$

In view of the relation $B^*_{n,M}B_{n,M}=A^*(T_nP_M)^*T_nP_MA$, it is enough to show that under the condition of the lemma we have $(T_nP_M)^*T_nP_M=P_M$.

It is easy to check that

$$\begin{aligned}(T_nP_M)^*T_nP_Mf(t)&=\sum_{k=0}^M\sum_{j=1}^{2k+1}\frac{1}{\rho^2}Y_{k,j}\left(\frac{t}{\rho}\right)\sum_{l=1}^n\omega_l^nY_{k,j}\left(\frac{\tau_l}{\rho}\right)\\&\quad\times\sum_{\nu=0}^M\sum_{\mu=1}^{2\nu+1}\frac{1}{\rho^2}Y_{\nu,\mu}\left(\frac{\tau_l}{\rho}\right)\left\langle Y_{\nu,\mu}\left(\frac{\cdot}{\rho}\right),f(\cdot)\right\rangle_{L^2(\Omega_\rho)}.\end{aligned} \tag{15}$$

Observe now that

$$\sum_{\nu=0}^M\sum_{\mu=1}^{2\nu+1}\frac{1}{\rho^2}Y_{\nu,\mu}\left(\frac{\tau_l}{\rho}\right)\left\langle Y_{\nu,\mu}\left(\frac{\cdot}{\rho}\right),f(\cdot)\right\rangle_{L^2(\Omega_\rho)}=P_Mf(\tau_l).$$

Keeping in mind that for $k=0,1,\dots,M,\ j=1,2,\dots,2k+1$ the function $Y_{k,j}(\frac{\tau}{\rho})P_Mf(\tau)$ is a spherical polynomial of degree not more than $2M$ and using the exactness of the formula (12) for such polynomials we can continue as follows

$$\begin{aligned}&\sum_{l=1}^n\omega_l^nY_{k,j}\left(\frac{\tau_l}{\rho}\right)\sum_{\nu=0}^M\sum_{\mu=1}^{2\nu+1}\frac{1}{\rho^2}Y_{\nu,\mu}\left(\frac{\tau_l}{\rho}\right)\left\langle Y_{\nu,\mu}\left(\frac{\cdot}{\rho}\right),f(\cdot)\right\rangle_{L^2(\Omega_\rho)}\\&=\sum_{l=1}^n\omega_l^nY_{k,j}\left(\frac{\tau_l}{\rho}\right)P_Mf(\tau_l)=\left\langle Y_{k,j}\left(\frac{\cdot}{\rho}\right),f(\cdot)\right\rangle_{L^2(\Omega_\rho)},\\&\quad k=0,1,\dots,M,\ \ j=1,2,\dots,2k+1.\end{aligned} \tag{16}$$

Combining upper Eqs. (15) and (16), we obtain the desired equality $(T_n P_M)^* T_n P_M = P_M$ that leads to the statement of the lemma. □

Remark 2.2 For n large enough, one can find a wide variety of the formulas (12) meeting the assumption of Lemma 2.1 [see, e.g., Xu (2003); Le Gia and Mhaskar (2006)].

3 Regularization error bounds

Now we return to the discussion of regularization of the Eqs. (7), (8). We assume that the initial Eq. (1) has a solution $x = x_*$. In view of the result by Mathé and Hofmann (2008), without loss of generality, we can assume that there exists an increasing function $\varphi : \left[0, \|A^*A\|\right] \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and the smoothness of the solution x_* is expressed in terms of the inclusion

$$x_* \in A_\varphi(H) = \left\{x : x = \varphi(A^*A)v,\ \|v\| \le H\right\}. \tag{17}$$

It means that in terms of (2), (3) the solution x_* admits the representation

$$x_*(\tau) = \sum_{i=0}^{\infty} \varphi(a_i^2) v_i(\tau) \left\langle v_i, v\right\rangle_{L^2(\Omega_R)},$$

where v is some function from $L^2(\Omega_R)$. In (17) the function φ is going under the name of an index function or smoothness index, and supposed to be unknown.

Recall that instead of (1) we are given (7), (8) that needs to be treated by means of regularization techniques, and we are going to use Tikhonov–Phillips method for such treatment. In this case, one can use the function $g_\alpha(t) = (\alpha + t)^{-1}$ that meets the inequalities

$$\begin{aligned} \sup_{0 \le t \le \|A^*A\|} |1 - t g_\alpha(t)| &\le 1, \\ \sup_{0 \le t \le \|A^*A\|} \sqrt{t}\, |g_\alpha(t)| &\le 1/(2\sqrt{\alpha}), \end{aligned} \tag{18}$$

to present the approximate solution (10) given by Tikhonov-Phillips method in the form $x_{\alpha,\delta} = g_\alpha(B_{n,M}^* B_{n,M}) B_{n,M}^* y^n$.

Then

$$\begin{aligned} x_* - x_{\alpha,\delta} &= x_* - g_\alpha(B_{n,M}^* B_{n,M}) B_{n,M}^* y^n \\ &= (I - g_\alpha(B_{n,M}^* B_{n,M}) B_{n,M}^* B_{n,M}) x_* - g_\alpha(B_{n,M}^* B_{n,M}) B_{n,M}^* y^n \\ &\quad + g_\alpha(B_{n,M}^* B_{n,M}) B_{n,M}^* B_{n,M} x_* \\ &= (I - g_\alpha(B_{n,M}^* B_{n,M}) B_{n,M}^* B_{n,M}) x_* + g_\alpha(B_{n,M}^* B_{n,M}) B_{n,M}^* (B_{n,M} x_* - y^n). \end{aligned} \tag{19}$$

Now we are going to estimate each term of (19). This will be done under the following assumption.

Assumption 3.1 The index function φ from (17) is operator monotone on the interval [0, 1] and such that φ^2 is concave.

Recall (Nair and Pereverzev 2007) that a function φ is operator monotone on the interval $J \subseteq [0, \infty)$ if for any pair of self-adjoint operators A, B with spectra in J, we have $\varphi(A) \leq \varphi(B)$ whenever $A \leq B$. As usual, the partial ordering $A \leq B$ for self-adjoint operators A, B on a Hilbert space X means that $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for any $x \in X$.

Remark 3.2 Note that in view of (3), (14) and Lemma 2.1 the spectra of the operators A^*A, $B^*_{n,M}B_{n,M}$ involved in our analysis are contained in [0, 1]. Therefore, it is not at all restrictive to assume that φ is defined on the interval [0, 1]. Moreover, in Pereverzev and Schock (1999) it has been shown that for the problem (1) the function $\varphi(u)$ in (17) cannot increase faster than $\log^{-\mu}\left(\frac{1}{u}\right)$, for some $\mu > 0$. The latter function is known to be operator monotone on [0, 1], and its square is a concave function. Therefore, in our context Assumption 3.1 does not pose any restriction.

Lemma 3.3 *Assume that the condition of Lemma 2.1 is satisfied, as well as (17) and Assumption 3.1. Then*

$$\left\|(I - g_\alpha(B^*_{n,M}B_{n,M})B^*_{n,M}B_{n,M})x_*\right\|_{L^2(\Omega_R)} \leq C\left[\varphi(\alpha) + \varphi\left(a^2_{(M+1)^2+1}\right)\right],$$

where $C > 0$ is some constant which does not depend on α, M, and φ.

The proof of Lemma 3.3 follows along the line of that of Lemma 1 by Nair and Pereverzev (2007), where one needs to use Lemma 2.1 instead of Proposition 1 from the mentioned paper. Therefore, we omit the proof of the result here.

Now we are going to derive an upper bound for the last term in (19). To obtain this bound we need to estimate the noise level in the Eq. (6). The latter one can be considered as a noisy version of the equation

$$T_n Ax = T_n y.$$

In view of (5), we have

$$\left\|T_n y - y^n\right\|_\omega = \delta := \left(\sum_{i=1}^{n} \omega_i^n \epsilon_i^2\right)^{1/2}, \tag{20}$$

where the quantity δ is calculated or estimated directly from the data errors ϵ_i and used as a measure of the noise level in the Eqs. (7), (8). Now the estimation of the last term in (19) is given by the following lemma.

Lemma 3.4 *Assume that the condition of Lemma 2.1 is satisfied, as well as (17) and Assumption 3.1. Then the next inequality holds true*

$$\left\| g_\alpha(B_{n,M}^* B_{n,M}) B_{n,M}^* (B_{n,M} x_* - y^n) \right\|_{L^2(\Omega_R)} \leq \frac{C_1 M^7}{\sqrt{\alpha} R\rho} \left(\frac{R}{\rho}\right)^{M^2} \varphi\left(a^2_{(M+1)^2+1}\right) + \frac{\delta}{2\sqrt{\alpha}},$$

where $C_1 > 0$ is some constant which does not depend on α, M, and φ.

Proof Using spectral calculus we may write

$$\begin{aligned} &\left\| g_\alpha(B_{n,M}^* B_{n,M}) B_{n,M}^* (B_{n,M} x_* - y^n) \right\|_{L^2(\Omega_R)} \\ &\leq \left\| g_\alpha(B_{n,M}^* B_{n,M}) B_{n,M}^* \right\|_{R^n_\omega \to L^2(\Omega_R)} \left\| B_{n,M} x_* - y^n \right\|_\omega \\ &\leq \frac{1}{2\sqrt{\alpha}} \left\| (B_{n,M} x_* - y^n) \right\|_\omega . \end{aligned}$$

The last norm can be bounded with the use of (20) as follows

$$\begin{aligned} \left\| B_{n,M} x_* - y^n \right\|_\omega &\leq \| T_n A Q_M x_* - T_n A x_* \|_\omega + \left\| T_n A x_* - y^n \right\|_\omega \\ &\leq \delta + \| T_n (A - A Q_M)(I - Q_M) x_* \|_\omega \\ &= \delta + \left(\sum_{i=1}^n \omega_i^n \, |(A - A Q_M)(I - Q_M) x_*(\tau_i)|^2 \right)^{1/2} \\ &\leq \delta + 2\rho\sqrt{\pi} \max_t (|(A - A Q_M)(I - Q_M) x_*(t)|). \end{aligned}$$

Moreover, we observe that

$$\begin{aligned} |(A - A Q_M)(I - Q_M) x_*(t)| &= \sum_{k=M+1}^{\infty} a_k \sum_{j=1}^{2k+1} \frac{1}{\rho} Y_{k,j}\left(\frac{t}{\rho}\right) \\ &\times \left\langle \frac{1}{R} Y_{k,j}\left(\frac{\cdot}{R}\right), (I - Q_M) x_*(\cdot) \right\rangle_{L^2(\Omega_R)} \\ &\leq \left[\sum_{k=M+1}^{\infty} \sum_{j=1}^{2k+1} \left(\frac{a_k}{\rho}\right)^2 Y^2_{k,j}\left(\frac{t}{\rho}\right) \right]^{1/2} \| (I - Q_M) x_* \|_{L^2(\Omega_R)} . \end{aligned}$$

Due to Proposition 2 from Mathé and Pereverzev (2003b) one can write

$$\begin{aligned} \| (I - Q_M) x_* \|_{L^2(\Omega_R)} &\leq H \left\| (I - Q_M) \varphi(A^* A) \right\|_{L^2(\Omega_R)} \\ &\leq H\varphi\left(\| A(I - Q_M) \|^2_{L^2(\Omega_R)} \right) \leq H\varphi\left(a^2_{(M+1)^2+1} \right). \end{aligned} \tag{21}$$

Now using (3) and the well-known inequality $\left|Y_{k,j}(\cdot)\right| \leq \sqrt{(2k+1)/4\pi}$ we may continue

$$\left[\sum_{k=M+1}^{\infty}\sum_{j=1}^{2k+1}\left(\frac{a_k}{\rho}\right)^2 Y_{k,j}^2\left(\frac{t}{\rho}\right)\right]^{1/2}$$
$$\leq \tilde{C}\left[\sum_{k=M+1}^{\infty}\left(\frac{R}{\rho}\right)^{2\sqrt{k}}\frac{k^2(2k+1)}{4\pi R^2\rho^4}\right]^{1/2}$$
$$\leq \frac{\tilde{C}_1}{2\sqrt{\pi}R\rho^2}\left[\int_{M+1}^{\infty}\left(\frac{R}{\rho}\right)^{2\sqrt{u}} u^3 du\right]^{1/2},$$
$$\leq \frac{\tilde{C}_2}{2\sqrt{\pi}R\rho^2}\left(\frac{R}{\rho}\right)^{M^2} M^7,$$

where $\tilde{C}$, $\tilde{C}_1$ and $\tilde{C}_2$ are some constants.

Hence,

$$\left\|B_{n,M}x_* - y^n\right\|_\omega \leq \frac{\tilde{C}_2 H}{R\rho}\left(\frac{R}{\rho}\right)^{M^2} M^7\varphi\left(a^2_{(M+1)^2+1}\right) + \delta, \tag{22}$$

and, finally, we obtain

$$\left\|g_\alpha(B^*_{n,M}B_{n,M})B^*_{n,M}(B_{n,M}x_* - y^n)\right\|_{L^2(\Omega_R)}$$
$$\leq \frac{C_1 M^7}{\sqrt{\alpha}R\rho}\left(\frac{R}{\rho}\right)^{M^2}\varphi\left(a^2_{(M+1)^2+1}\right) + \frac{\delta}{2\sqrt{\alpha}}.$$

□

Both Lemma 3.3 and Lemma 3.4 provide us with the following error bound.

$$\left\|x_{\alpha,\delta} - x_*\right\|_{L^2(\Omega_R)} \leq C\left[\varphi(\alpha) + \varphi\left(a^2_{(M+1)^2+1}\right)\right]$$
$$+\frac{C_1 M^7}{\sqrt{\alpha}R\rho}\left(\frac{R}{\rho}\right)^{M^2}\varphi\left(a^2_{(M+1)^2+1}\right) + \frac{\delta}{2\sqrt{\alpha}}. \tag{23}$$

Remark 3.5 Note that in the context of gravity gradiometry it is expected that in (5) the data errors ϵ_i are of order $10^{-9}(s^{-2})$ [see, e.g., Freeden (1999), Appendix B]. Moreover, using the cubature formula (12) it is obvious that the weights ω_i^n are of order 10^{12}. If so, the quantity δ^2 from (20) is of order 10^{-6}, and using (3) it is easy to check that for any M the values of the quantities $a^2_{(M+1)^2+1}$ and $\frac{M^7}{R\rho}(\frac{R}{\rho})^{M^2}\varphi(a^2_{(M+1)^2+1})$ do not exceed the value of such δ^2. On the other hand, from (23) it is clear that to guarantee a reasonable approximation one has to choose α such that $\delta/\sqrt{\alpha} < 1$ or, which is the same thing as $\alpha > \delta^2$.

Thus, in the present context the error bound (23) can be reduced to the following one

$$\left\| x_{\alpha,\delta} - x_* \right\|_{L^2(\Omega_R)} \leq C_2 \left(\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}} \right), \tag{24}$$

where the value of C_2 does not depend on M, α and φ.

Following Mathé and Pereverzev (2003a), we consider now the function $\theta(\alpha) = \varphi(\alpha)\sqrt{\alpha}$. Then the error bound (24) tells us that a choice of $\alpha = \alpha_{opt} = \theta^{-1}(\delta)$, which balances $\varphi(\alpha)$ with $\delta/\sqrt{\alpha}$, leads to an accuracy of order $\varphi(\theta^{-1}(\delta))$, which is optimal with respect to δ. Unfortunately, an a priori parameter choice $\alpha = \alpha_{opt}$ can seldom be used in practice because the smoothness properties of the unknown solution x_* reflected in the function φ are generally unknown. On the other hand, the order of accuracy $O(\varphi(\theta^{-1}(\delta)))$ can be considered as a benchmark for a posteriori parameter choice strategies. In the next section we focus on the analysis of an adaptative parameter choice rule that allows to achieve the optimal order of accuracy (benchmark) without any a priori knowledge on the unknown solution.

4 A choice of the regularization parameter

As it has been mentioned in Mathé and Pereverzev (2006b), among the variety of known a posteriori parameter choice stategies there are two methods which enjoy the advantage of implementation simplicity. One of them is well-known discrepancy principle (DP) and another one is general adaptation strategy (GAS), also known as Lepskii-type balancing principle [Lu and Pereverzev (2013), Section 1.1].

Assume we are given an increasing geometric sequence of parameter values $\{\alpha_i\}_{i=1}^n$. DP begins from the largest α_n and takes smaller and smaller values $\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_k$ until, for example, $\left\| B_{n,M} x_{\alpha,\delta} - y^n \right\|_\omega \leq \tau\delta$, where τ is a design parameter, which is at least two. Then the first satisfying value $\alpha = \alpha_k$ is the regularization parameter of our choice.

At the same time, GAS operates in the opposite direction. Here we start with the smallest value α_1 and take larger and larger values of the parameters $\alpha_2, \alpha_3, \ldots, \alpha_j, \alpha_{j+1}$ until $\left\| x_{\alpha_j,\delta} - x_{\alpha_{j+1},\delta} \right\| > \tau\delta/\sqrt{\alpha_{j+1}}$. Then GAS suggests the choice $\alpha = \alpha_j$.

Thus, when applying GAS we start with the hardest regularized problem (10), while DP allows us to begin with the easiest one, and move to the harder one only if necessary.

However, it is known that DP does not provide the best order of accuracy for all ill-posed problems for which Tikhonov–Phillips regularization allows us to obtain the best order of reconstruction. Specifically, the best possible error of Tikhonov–Phillips regularization is $O(\delta^{2/3})$, while in combination with the DP one can achieve the accuracy $O(\delta^{1/2})$ at best due to the saturation effect. On the other hand, GAS is a rule that allows us to reach the best order of accuracy for all problems that in principle can be treated in an optimal way by Tikhonov–Phillips regularization.

In our case, it is necessary to remind that the problem (1) is believed to be severely ill-posed. For such a problem an order of accuracy better than $O(\delta^{1/2})$ cannot be reached in general (Pereverzev and Schock 2000). Therefore, it is natural to expect that DP will not suffer from the saturation for our problem. Thus, the use of DP within the framework of Tikhonov–Phillips regularization is reasonable for us.

As it was mentioned, DP can provide us with the best order of accuracy in the case of severely ill-posed problems when Tikhonov–Phillips regularization is applied. However, to the best of our knowledge, in the literature this statement has been proven only for the case, when the data y in (1) are given in the image space of the operator A. In our situation, the data are given not in L^2 space, but in $\mathbb{R}^n$, and in this section we are going to prove that for this case DP also provides us with the best order of accuracy indicated in the inequality (24).

Assume that using DP we have chosen the regularization parameter $\alpha = \alpha_* = \alpha_k$. In the present context this parameter α_* satisfies the following inequalities

$$\left\| B_{n,M} g_{\alpha_*}(B^*_{n,M} B_{n,M}) B^*_{n,M} y^n - y^n \right\|_\omega \geq \tau_1 \delta, \tag{25}$$

$$\left\| B_{n,M} g_{\alpha_*}(B^*_{n,M} B_{n,M}) B^*_{n,M} y^n - y^n \right\|_\omega \leq \tau \delta, \tag{26}$$

where τ, τ_1 are some numbers such that $2 < \tau_1 < \tau$. The following theorem states that for the regularization parameter α_* Tikhonov–Phillips regularization provides us with the solution $x_{\alpha_*,\delta}$, realizing an accuracy of optimal order.

Theorem 4.1 *Assume that the conditions of Lemma 3.4 are satisfied. If $\alpha = \alpha_*$ is chosen according to (25), (26) then*

$$\left\| x_* - x_{\alpha_*,\delta} \right\|_{L^2(\Omega_R)} \leq C_3 \varphi(\theta^{-1}(\delta)),$$

where $C_3 > 0$ is a constant which does not depend on M, α_*, and φ.

Proof We prove the theorem by considering two cases $\alpha_* \geq \alpha_{opt}$ and $\alpha_* < \alpha_{opt}$. Let us first assume that $\alpha_* \geq \alpha_{opt}$. It is easy to check that

$$\begin{aligned} & B_{n,M}(I - g_{\alpha_*}(B^*_{n,M} B_{n,M}) B^*_{n,M} B_{n,M}) x_* \\ & \quad = (I - B_{n,M} g_{\alpha_*}(B^*_{n,M} B_{n,M}) B^*_{n,M}) y^n + (I - g_{\alpha_*}(B_{n,M} B^*_{n,M}) B_{n,M} B^*_{n,M}) \\ & \qquad \times (B_{n,M} x_* - y^n). \end{aligned}$$

In view of (26), (18), (22), and Remark 3.5 we have

$$\begin{aligned} & \left\| B_{n,M}(I - g_{\alpha_*}(B^*_{n,M} B_{n,M}) B^*_{n,M} B_{n,M}) x_* \right\|_\omega \\ & \quad \leq \left\| (I - B_{n,M} g_{\alpha_*}(B^*_{n,M} B_{n,M}) B^*_{n,M}) y^n \right\|_\omega + \left\| B_{n,M} x_* - y^n \right\|_\omega \leq \tau\delta + 2\delta. \end{aligned} \tag{27}$$

Moreover, from (19), Lemma 3.4, and Remark 3.5 we have a bound

$$
\begin{aligned}
&\left\| x_* - g_{\alpha_*}(B^*_{n,M} B_{n,M}) B^*_{n,M} y^n \right\|_{L^2(\Omega_R)} \\
&\leq \left\| (I - g_{\alpha_*}(B^*_{n,M} B_{n,M}) B^*_{n,M} B_{n,M}) x_* \right\|_{L^2(\Omega_R)} + \frac{C_2 \delta}{\sqrt{\alpha_{opt}}}.
\end{aligned} \tag{28}
$$

From (21) and Remark 3.5 it follows that

$$
\begin{aligned}
&\left\| (I - g_{\alpha_*}(B^*_{n,M} B_{n,M}) B^*_{n,M} B_{n,M}) x_* \right\|_{L^2(\Omega_R)} \\
&\leq \| x_* - Q_M x_* \|_{L^2(\Omega_R)} + \left\| Q_M x_* - g_{\alpha_*}(B^*_{n,M} B_{n,M}) B^*_{n,M} B_{n,M} x_* \right\|_{L^2(\Omega_R)} \\
&\leq H\varphi(\alpha_{opt}) + \left\| Q_M x_* - g_{\alpha_*}(B^*_{n,M} B_{n,M}) B^*_{n,M} B_{n,M} x_* \right\|_{L^2(\Omega_R)}.
\end{aligned} \tag{29}
$$

In view of the decomposition (2) we may continue

$$
\begin{aligned}
&\left\| Q_M x_* - g_{\alpha_*}(B^*_{n,M} B_{n,M}) B^*_{n,M} B_{n,M} x_* \right\|^2_{L^2(\Omega_R)} \\
&= \sum_{i=0}^{(M+1)^2} \langle v_i, x_* \rangle^2_{L^2(\Omega_R)} (1 - g_{\alpha_*}(b_i) b_i)^2 = \sum_{i=0}^{(M+1)^2} \langle v_i, x_* \rangle^2_{L^2(\Omega_R)} \left(\frac{\alpha_*}{\alpha_* + b_i} \right)^2,
\end{aligned}
$$

where b_i are singular values of the operator $B^*_{n,M} B_{n,M}$.

Note also that for $\alpha_* \geq \alpha_{opt}$ it holds that

$$
\left(\frac{\alpha_*}{\alpha_* + t} \right)^2 \leq 4 \left(\frac{\alpha_{opt}}{\alpha_{opt} + t} \right)^2 + \frac{t}{\alpha_{opt}} \left(\frac{\alpha_*}{\alpha_* + t} \right)^2. \tag{30}
$$

Using (30), Lemma 3.3, Remark 3.5, and (27) we obtain

$$
\begin{aligned}
&\sum_{i=0}^{(M+1)^2} \langle v_i, x_* \rangle^2_{L^2(\Omega_R)} \left(\frac{\alpha_*}{\alpha_* + b_i} \right)^2 \leq 4 \sum_{i=0}^{(M+1)^2} \langle v_i, x_* \rangle^2_{L^2(\Omega_R)} \left(\frac{\alpha_{opt}}{\alpha_{opt} + b_i} \right)^2 \\
&\quad + \sum_{i=0}^{(M+1)^2} \langle v_i, x_* \rangle^2_{L^2(\Omega_R)} \frac{b_i}{\alpha_{opt}} \left(\frac{\alpha_*}{\alpha_* + b_i} \right)^2 \\
&\leq 4 \left\| (I - g_{\alpha_{opt}}(B^*_{n,M} B_{n,M}) B^*_{n,M} B_{n,M}) x_* \right\|^2_{L^2(\Omega_R)} \\
&\quad + \frac{1}{\alpha_{opt}} \left\| (B^*_{n,M} B_{n,M})^{1/2} (I - g_{\alpha_*}(B^*_{n,M} B_{n,M}) B^*_{n,M} B_{n,M}) x_* \right\|^2_{L^2(\Omega_R)} \\
&\leq 4 \left[C_2 \varphi(\alpha_{opt}) \right]^2 + \frac{1}{\alpha_{opt}} \left\| B_{n,M} (I - g_{\alpha_*}(B^*_{n,M} B_{n,M}) B^*_{n,M} B_{n,M}) x_* \right\|^2_{\omega} \\
&\leq 4 \left[C_2 \varphi(\alpha_{opt}) \right]^2 + \frac{\delta^2}{\alpha_{opt}} (\tau + 2)^2 \leq \varphi(\alpha_{opt})^2 (2C_2 + \tau + 2)^2.
\end{aligned}
$$

Combining this bound with (29) and (28), we finally obtain

$$\begin{aligned}&\left\|x_* - g_{\alpha_*}(B_{n,M}^* B_{n,M}) B_{n,M}^* y^n\right\|_{L^2(\Omega_R)} \leq \varphi(\alpha_{opt})\,[3C_2 + \tau + 2 + H]\\&\quad \leq C_3\varphi(\alpha_{opt}) = C_3\varphi(\theta^{-1}(\delta)).\end{aligned}$$

Consider now the case $\alpha_* < \alpha_{opt}$. Recall that under our assumption, φ^2 is concave. From this it follows that $\sqrt{t}/\varphi(t)$ is non-decreasing. Indeed, if $t_1 < t_2$ are some positive numbers then for a concave function φ^2 we may write

$$\begin{aligned}\varphi^2(t_1) &= \varphi^2\left(t_2\frac{t_1}{t_2}\right) = \varphi^2\left(t_2\frac{t_1}{t_2} + \left(1 - \frac{t_1}{t_2}\right)\cdot 0\right)\\&\geq \left(\frac{t_1}{t_2}\right)\varphi^2(t_2) + \left(1 - \frac{t_1}{t_2}\right)\varphi^2(0) = \frac{t_1}{t_2}\varphi^2(t_2).\end{aligned}$$

This means that $t_1/\varphi^2(t_1) \leq t_2/\varphi^2(t_2)$, when $t_1 < t_2$ or which is the same thing as $\sqrt{t}/\varphi(t)$ is non-decreasing.

Now we employ the fact [see, e.g., Lu and Pereverzev (2013), Proposition 2.7] that for any non-decreasing function $\psi : [0, \infty) \to [0, \infty)$ such that $t/\psi(t)$ is also non-decreasing it holds

$$\sup_t |(1 - t g_\alpha(t))\psi(t)| \leq \psi(\alpha). \tag{31}$$

Using Lemma 2.1 and the inequality (31) with $\psi(t) = \sqrt{t}\varphi(t)$ and $\psi(t) = \sqrt{t}$, we obtain

$$\begin{aligned}&\left\|B_{n,M}(I - g_{\alpha_*}(B_{n,M}^* B_{n,M}) B_{n,M}^* B_{n,M}) x_*\right\|_\omega\\&\quad \leq \left\|B_{n,M}(I - g_{\alpha_*}(B_{n,M}^* B_{n,M}) B_{n,M}^* B_{n,M})\varphi(B_{n,M}^* B_{n,M}) v\right\|_\omega\\&\qquad + \left\|B_{n,M}(I - g_{\alpha_*}(B_{n,M}^* B_{n,M}) B_{n,M}^* B_{n,M})(\varphi(A^*A) - \varphi(B_{n,M}^* B_{n,M})) v\right\|_\omega\\&\quad \leq \left\|(B_{n,M}^* B_{n,M})^{1/2}(I - g_{\alpha_*}(B_{n,M}^* B_{n,M}) B_{n,M}^* B_{n,M})\varphi(B_{n,M}^* B_{n,M}) v\right\|_{L^2(\Omega_R)}\\&\qquad + \left\|(B_{n,M}^* B_{n,M})^{1/2}(I - g_{\alpha_*}(B_{n,M}^* B_{n,M}) B_{n,M}^* B_{n,M})\right\|_{L^2(\Omega_R)}\\&\qquad \times \left\|(\varphi(A^*A) - \varphi(B_{n,M}^* B_{n,M})) v\right\|_{L^2(\Omega_R)} \leq H\sqrt{\alpha_*}(\varphi(\alpha_*) + \varphi(a^2_{(M+1)^2+1})).\end{aligned}$$

Then in view of (25), (22), and Remark 3.5, we can continue

$$\begin{aligned}2H\sqrt{\alpha_*}\varphi(\alpha_*) &\geq \left\|B_{n,M}(I - g_{\alpha_*}(B_{n,M}^* B_{n,M}) B_{n,M}^* B_{n,M}) x_*\right\|_\omega\\&\geq \left\|(I - B_{n,M} g_{\alpha_*}(B_{n,M}^* B_{n,M}) B_{n,M}^*) y^n\right\|_\omega - \left\|B_{n,M} x_* - y^n\right\|_\omega \geq \tau_1\delta - 2\delta.\end{aligned}$$

Thus,

$$\frac{\delta}{\sqrt{\alpha_*}} \leq \frac{2H}{\tau_1 - 2}\varphi(\alpha_*) \leq \frac{2H}{\tau_1 - 2}\varphi(\alpha_{opt}).$$

Combining this inequality with (24), where $\alpha = \alpha_* < \alpha_{opt}$, we finally arrive at the bound that proves the statement of the theorem

$$\left\|x_* - g_{\alpha_*}(B_{n,M}^* B_{n,M}) B_{n,M}^* y^n\right\|_{L^2(\Omega_R)} \leq C_2 \varphi(\alpha_{opt}) \left(1 + \frac{2H}{\tau_1 - 2}\right)$$
$$\leq C_3 \varphi(\alpha_{opt}) = C_3 \varphi(\theta^{-1}(\delta)).$$

□

5 Numerical examples

In this section we are going to present some numerical experiments to verify the analysis from the previous sections. Similar to Xu et al. (2006) in our experiments we do not work with real data but do with artificially generated ones. In our tests we consider the case when one is interested in reconstruction of the Fourier coefficients with respect to the spherical harmonics up to degree $M = 50$, i.e., the total amount of the Fourier coefficients to be reconstructed is $N = (M + 1)^2 = 2601$.

We follow Bauer et al. (2007); Lu and Pereverzev (2010) and assume that the orbit height is 400 km. In this case the decay character of the singular values of the operator A is modeled as

$$a_i = (1.06)^{-i}, \ i = 0, 1, \ldots, M.$$

Note that such singular values lead us to a more ill-posed inverse problem than the original one described by (2) and (3). Therefore, we can expect that our algorithm will work well in real applications if it works in the experiments under consideration.

Our analysis from the previous sections shows that to reconstruct $N = (M + 1)^2$ Fourier coefficients we need to satisfy the condition of Lemma 2.1. From the literature it is known [see, Graf et al. (2009), Remark 2.2, for example] that the exactness of a cubature formula for spherical polynomials up to degree $2M$ can be achieved only if it has at least $n = (M+1)^2$ nodes. But the positiveness of the weights is not guaranteed in this situation, and in practice one needs to take the amount n of collocation points which is much larger than $(M + 1)^2$ to obtain the positive weights.

For our method, we will also follow Graf et al. (2009) and consider Gauss-Legendre quadrature grid of collocation points where the positive quadrature weights are given analytically. The number of nodes in this case is $n = 2(M+1)^2$ and the corresponding cubature formula is exact for all spherical polynomials of degree $2M$ as it is required by Lemma 2.1.

The position of every node on the sphere is defined by a pair $(\theta_i; \phi_j), \ i = 0, \ldots, M, \ j = 0, \ldots, 2M + 1$, where $\theta_i \in [0, \pi)$ is the longitudinal direction and $\phi_j \in [0, 2\pi)$ is the latitudinal one. For the grid used in our experiments latitudinal coordinates of the collocation points are $\phi_j = j\pi/(M + 1), \ j = 0, \ldots, 2M + 1$ and longitudinal coordinates θ_i are the arc-cosines of the zeros of Legendre polynomial of degree $M + 1$. The grid of the collocation points is shown on Fig. 1.

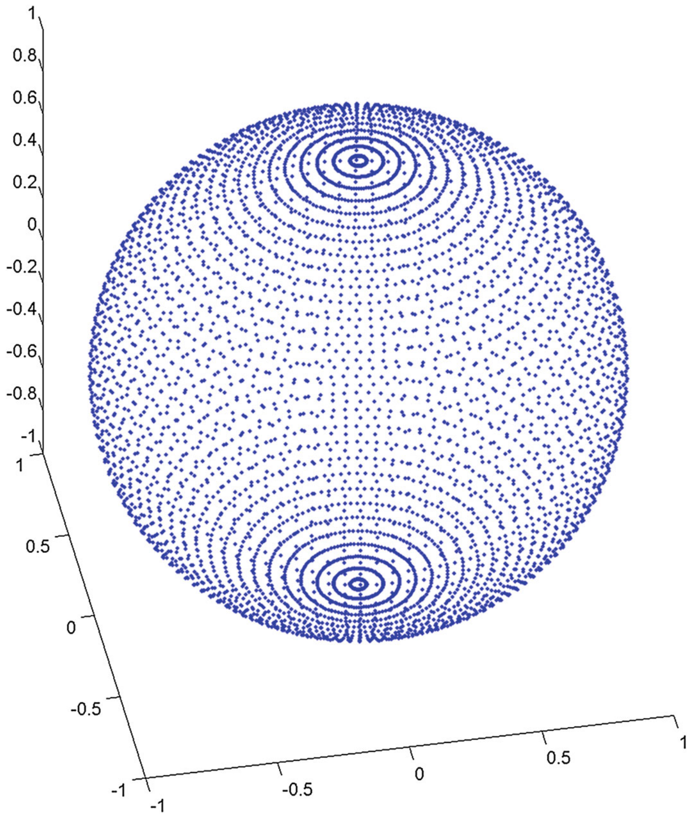

Fig. 1 Gauss–Legendre quadrature grid of 51×102 collocation points on the unit sphere

At the same time, in attempt to reconstruct $(M+1)^2$ Fourier coefficients from the amount of data smaller than $n = 2(M+1)^2$ one can try to use the following approach, which is, at first, estimate $(M+1)^2$ Fourier coefficients of the right-hand side y from the available amount of collocation data and then apply the regularized projection scheme, as it was mentioned in the introduction.

Theoretically, to estimate $(M+1)^2$ Fourier coefficients of the right-hand side y we need the same amount n of collocation data. Then we can solve an interpolation problem which has a unique solution. However, as it was mentioned in Keiner et al. (2007), such interpolation problem may lead to an ill-conditioned matrix. To avoid this, it is reasonable to consider the overdetermined case $n > (M+1)^2$ and solve the corresponding least-squares problem. In our experiments we consider the case when the amount of collocation data is in a range $(M+1)^2 < n \leq 2(M+1)^2$.

We present results of numerical experiments which show that the amount of collocation data $n = 2(M+1)^2 = 5202$ cannot be essentially reduced without a loss of accuracy. To estimate the performance of our method, we conduct the experiments in the following way. First, we generate some "exact solution" x_* of our problem (namely, N Fourier coefficients), then construct the operator $B_{n,M}$, which we apply to

Table 1 Comparison of the regularized collocation and projection (after preprocessing) methods

n_1	n_2	$\max(\epsilon_i)$	α_*	$\|x_{\alpha,\delta} - x_*\| / \|x_*\|$
51	102	3.4887e−10	1.1790e−05	0.0019
101	101	3.0613e−10	9.5500e−06	0.0016
51	102	2.9917e−10	1.3100e−05	0.0020
50	102	3.0192e−10	1.3100e−05	0.0171
50	102	2.8035e−10	1.3100e−05	0.0377
51	100	2.9387e−10	1.3100e−05	0.0107
51	52	2.9882e−10	1.3100e−05	1.0123
51	51	3.2154e−10	1.3100e−05	1.0621

The last column contains the values of the relative error of the collocation method (the first two rows) and the projection method (the other rows) after least-squares prepossessing of collocation data
n_1 Number of latitude points, n_2 number of longitude points, $\max(\epsilon_i)$ maximal data error corresponding to noise level δ of 1 %, α_* chosen regularization parameter

x_* to obtain "satellite data" and, at the last stage of data generation, we spoil them by adding a random noise. Finally, we reconstruct N Fourier coefficients from these noisy data and compare the obtained regularized solution with the exact one. The Fourier coefficients of the "exact solution" are uniformly distributed random values on the interval $(-1, 1)$.

Table 1 shows the performance of the regularized collocation method and the regularized projection method based on different amount of the collocation data. To use the projection scheme, the data were preprocessed by the least-squares method, as it was described above. It is worth to mention that we also tried the weighted least-squares method, as in Keiner et al. (2007), but for our problem this method did not provide us with better results than the classical least-squares method with unit weights.

From Table 1 one can conclude that the number of collocation points, which guarantees the exactness of the cubature formula with positive weights for spherical polynomials up to degree $2M$, plays a crucial role in order to obtain a good reconstruction of the spherical Fourier coefficients of the solution up to degree M. This conclusion is in agreement with our theoretical analysis.

Acknowledgments The authors are supported by the Austrian Fonds Zur Forderung der Wissenschaftlichen Forschung (FWF), Grant P25424.

References

Bauer, F., Mathé, P., Pereverzev, S.V.: Local solutions to inverse problems in geodesy, the impact of the noise covariance structure upon the accuracy of estimation. J. Geod. **81**, 39–51 (2007)
Engl, H.W., Hanke, M., Neubauer, A.: Regularization of Inverse Problems. Kluwer, Dordrecht (1996)
Freeden, W., Pereverzev, S.: Spherical Tikhonov regularization wavelets in satellite gravity gradiometry with random noise. J. Geod. **74**, 730–736 (2001)
Freeden, W.: Multiscale Modeling of Spaceborne Geodata. Teubner, Stuttgart (1999)
Graf, M., Kunis, S., Potts, D.: On the computation of nonnegative quadrature weights on the sphere. Appl. Comput. Harm. Anal. **27**, 124–132 (2009)

Groetsch, C.W.: Convergence analysis of a regularized degenerate kernel method for Fredholm integral equations of the first kind. Integr. Eqns. Oper. Theory **13**, 67–75 (1999)

Keiner, J., Kunis, S., Potts, D.: Efficient reconstruction of functions on the sphere from scattered data. J. Fourier Anal. Appl. **13**, 435–458 (2007)

Le Gia, Q.T., Mhaskar, H.N.: Polynomial operators and local approximation of solutions of pseudo-differential equations on the sphere. Numer. Math. **103**, 299–322 (2006)

Le Gia, Q.T., Mhaskar, H.N.: Localized linear polynomial operators and quadrature formulas on the sphere. SIAM J. Numer. Anal. **47**(1), 440–466 (2008)

Lu, S., Pereverzev, S.V.: Regularization Theory for Ill-Posed Problems Selected topics. Walter de Gruyter GmbH, Boston (2013)

Lu, S., Pereverzev, S.V.: Multiparameter regularization in downward continuation of satellite data. In: Handbook of Geomathematics, vol 2, pp. 813–832. Springer, Heidelberg (2010)

Mathé, P., Hofmann, B.: How general are general source conditions? Inverse Probl. **24**, 015009 (2008)

Mathé, P., Pereverzev, S.V.: Moduli of continuity for operator valued functions. Numer. Funct. Anal. Optim. **23**(56), 623–631 (2002)

Mathé, P., Pereverzev, S.V.: Geometry of linear ill-posed problems in variable Hilbert scales. Inverse Probl. **19**(3), 789–803 (2003a)

Mathé, P., Pereverzev, S.V.: Discretization strategy for linear ill-posed problems in variable Hilbert scales. Inverse Probl. **19**(6), 1263–1277 (2003b)

Mathé, P., Pereverzev, S.V.: Regularization of some linear inverse problems with discretized random noisy data. Math. Comput. **75**, 1913–1929 (2006a)

Mathé, P., Pereverzev, S.V.: The discretized discrepancy principle under general source conditions. J. Complex. **22**, 371–381 (2006b)

Nair, M.T., Pereverzev, S.V.: Regularized collocation method for Fredholm integral equations of the first kind. J. Complex. **23**(46), 454–467 (2007)

Pavlis, N.K., Holmes, S.A., Kenyon, S.C., Factor, J.K.: An earth gravitational model to degree 2160: EGM2008. EGU General, Assembly, pp. 13–18 (2008)

Pereverzev, S.V., Schock, E.: Error estimates for band-limited spherical regularization wavelets in an inverse problem of satellite geodesy. Inverse Probl. **15**, 881–890 (1999)

Pereverzev, S.V., Schock, E.: Morozov's discrepancy principle for Tikhonov regularization of severely ill-posed problems in finite-dimentional subspaces. Numer. Func. Anal. Optimi. **21**, 901–916 (2000)

Pietsch, A.: Operator Ideals. North-Holland, Amsterdam (1980)

Plato, R., Vainikko, G.: On the regularization of projection methods for solving ill-posed problems. Numer. Math. **57**, 63–70 (1990)

Rajan, M.P.: Convergence analysis of a regularized approximation for solving Fredholm integral equation of the first kind. J. Math. Anal. Appl. **279**, 522–530 (2003)

Rebhan, H., Aguirre, M., Johannessen, J.: The gravity field and steady-state ocean circulaton explorer mission - GOCE. ESA Earth Obs. Q. **66**, 6–11 (2000)

Tapley, B., Ries, J., Bettadpur, S., Chambers, D., Cheng, M., Condi, F., Gunter, B., Kang, Z., Nagel, P., Pastor, R., Pekker, T., Poole, S., Wang, F.: GGM02-an improved earth gravity field model from GRACE. J. Geod. **79**(8), 467–478 (2005)

Tikhonov, A.N.: On the regularization of ill-posed problems. Dokl. Akad. Nauk SSSR **153**, 49–52 (1963)

Xu, P.L., Fukuda, Y., Liu, Y.M.: Multiple parameter regularization: numerical solutions and applications to determination of geopotential from precise satellite orbits. J. Geod. **80**, 17–27 (2006)

Xu, Y.: Polynomial interpolation on the unit sphere. SIAM J. Numer. Anal. **41**, 751–766 (2003)